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NOTE

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THE INTERPRETATION OF TWO SYSTEMS OF MODAL LOGIC

A. N. PRIOR

Between the system set out by Łukasiewicz in his recent paper "A System of Modal Logic"[1], and the modal systems earlier developed by such writers as Lewis and von Wright, there are a number of striking differences, to some of which Łukasiewicz himself draws attention (particularly in Section 9 of his paper). I shall suggest here that these differences are due simply to the fact that what Łukasiewicz on the one hand and Lewis and von Wright on the other mean by "Possibly" are two quite different things, and I shall offer a pair of simple interpretations which will adequately account for the peculiarities of each. As both of these interpretations fall within the two-valued propositional calculus with functorial variables and with quantifiers, I shall contend that multi-valued systems are not required for the interpretation of the "Possibly" which occurs in the systems under consideration. This is not to deny that there are yet other senses of "Possibly" for which a logic of more than two truth-values is required. It is in fact my belief that there are such other senses of "Possibly."[2]

1. THE INTERPRETATION OF THE LEWIS-VON WRIGHT MODALITIES AS QUANTIFICATIONS. – It is fairly clear that the intention behind the systems of Lewis and von Wright is that "Possibly p" should be equivalent to something like "The truth of p would not violate any logical law", "Necessarily p" being similarly equivalent to something like "The falsehood of p would violate some logical law." The features of these systems to which Łukasiewicz refers follow readily from this intention. For example, there are clearly cases in which the truth of p would not violate any logical law, and the truth of Np would not violate any logical law either ("All men are mortal" and "Not all men are mortal" would do as illustrations of this), but the truth of KpNp would always violate a logical law. Hence Lewis's objection to the the formula $CK\Delta p\Delta Np\Delta KpNp$, and his consequent objection to $CK\Delta p\Delta q\Delta Kpq$, from which, if it were a thesis, the former would follow by the substitution q/Np. Again, von Wright's rule 9.2, "$\vdash \alpha \rightarrow \vdash \Gamma\alpha$", seems perfectly natural if "$\Gamma\alpha$" is taken as meaning that the truth of α is guaranteed by some logical law.

The following interpretation of the Lewis-von Wright modal calculi within the extended two-valued propositional calculi of Łukasiewicz and Meredith would be in accord with this fundamental intention: Read the propositional variables "p", "q", "r" etc. of the modal calculi as "δp", "γp", "ϵp", etc.; read

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"Δ" as "Σp", and read "Γ" as "Πp".[3] The rules and axioms of von Wright's strongest system M" then receive the following translations:

9.1 $\vdash E\alpha\beta \rightarrow \vdash E\Delta\alpha\Delta\beta$ becomes $\vdash E\alpha\beta \rightarrow \vdash E\Sigma p\alpha\Sigma p\beta$

9.2 $\vdash \alpha \rightarrow \vdash \Gamma\alpha$ becomes $\vdash \alpha \rightarrow \vdash \Pi p\alpha$

9.3 $Cp\Delta p$ becomes $C\delta p\Sigma p\delta p$

9.4 $E\Delta ApqA\Delta p\Delta q$ becomes $E\Sigma pA\delta p\gamma pA\Sigma p\delta p\Sigma p\gamma p$

9.5 $C\Delta\Delta p\Delta p$ becomes $C\Sigma p\Sigma p\delta p\Sigma p\delta p$

9.6 $C\Delta N\Delta pN\Delta p$ becomes $C\Sigma pN\Sigma p\delta pN\Sigma p\delta p$.

All of these "translations" are provable in the two-valued propositional C-N-δ calculus with the single axiom $C\delta pC\delta Np\delta q$, supplemented by Łukasiewicz's rules $\Pi 1$, $\Pi 2$, $\Sigma 1$ and $\Sigma 2$ for the introduction of quantifiers. We may therefore regard this system as simply an incomplete fragment of this C-N-δ-Π calculus. At least, we may regard it thus if we may prove in this calculus the counterparts not only of 9.1 – 9.6 but also of the remainder of von Wright's basis for M" – a set of axioms and rules for the ordinary propositional calculus, and the permission to substitute formulae constructed by means of "Δ", as well as formulae constructed by means of "N" and "C", for propositional variables in theses.

As to these last points, it is clear that the usual axioms and rules for the two-valued calculus in "C" and "N" will continue to hold when the variables "p", "q", "r", etc. are systematically replaced by "δp", "γp", "ϵp", etc. The extension of the rule of substitution to cover such substitutions as $p/\Delta q$ presents a little more difficulty. The substitution $p/\Delta q$ would be turned by our system of translations into $\delta p/\Sigma p\gamma p$, and such substitutions are not directly covered by ordinary δ-substitution. They are justifiable, however, by derivative rules. For in the C-N-δ-Π system we have as a thesis the assertion $E\Sigma p\gamma pA\gamma p\gamma Np$, so that the substitution $\delta p/\Sigma p\gamma p$ is justified if the substitution $\delta/A\gamma'\gamma N'$ is justified, and this is a regular δ-substitution. More complicated substitutions involving "Δ" may be justified by using more complicated equivalences derivable from $E\Sigma p\gamma pA\gamma p\gamma Np$.

This system of translation, it may be noted, will turn the disputed formula $CK\Delta p\Delta q\Delta Kpq$ into $CK\Sigma p\delta p\Sigma p\gamma p\Sigma pK\delta p\gamma p$, and this may be rejected by an argument very similar in form to that by which Lewis (as quoted by Łukasiewicz) rejects $CK\Delta p\Delta q\Delta Kpq$. For if the formula

$$CK\Sigma p\delta p\Sigma p\gamma p\Sigma pK\delta p\gamma p$$

were a thesis, we could derive from it (in the full C-N-δ-Π calculus) by the substitutions $\delta/'$ and γ/N the further thesis

$$CK\Sigma pp\Sigma pNp\Sigma pKpNp.$$

But here Σpp and ΣpNp, and consequently the antecedent $K\Sigma pp\Sigma pNp$, are provable formulae, while the consequent $\Sigma pKpNp$ is demonstrably false. Again, the "strong" principle of extensionality $CCpqC\Delta p\Delta q$, which Łukasiewicz accepts and von Wright rejects, becomes on translation

$$CC\delta p\gamma pC\Sigma p\delta p\Sigma p\gamma p.$$

If this were a thesis, the substitutions p/Cpp, δ/N, γ/K'N' would give us as a further thesis

$$CCNCppKCppNCppC\Sigma pNp\Sigma pKpNp.$$

Here the first antecedent CNCppKCppNCpp is demonstrably true (both its antecedent NCpp and its consequent KCppNCpp being demonstrably false), and the second antecedent ΣpNp is also demonstrably true, but the final consequent $\Sigma pKpNp$ is demonstrably false. Hence the thesis $CCpqC\Delta p\Delta q$ must be rejected if "p", "q" and "Δ" are given our present interpretations.

The fact that in the Lewis-von Wright systems there is no means of disproving the formula $C\Delta pp$ simply reflects the fragmentary character of these systems. On the view now being put forward, although the formula is not disprovable in the Lewis-von Wright systems themselves, it is disprovable in the complete system of which they are fragments, for the formula $C\Sigma p\delta p\delta p$, if asserted as a thesis, is demonstrably inconsistent with the axiom and rules of the complete C-N-δ-Π system. (Perform in $C\Sigma p\delta p\delta p$ the substitutions p/Cpp and δ/N, and we obtain $C\Sigma pNpNCpp$, in which the antecedent is demonstrably true and the consequent demonstrably false. This proof cannot be formalised within the Lewis-von Wright fragments, as the formula $C\Sigma pNpNCpp$, with no "δ" or "γ", is not a translation of any modal formula.)

The incompleteness of the Lewis-von Wright systems is in fact strictly comparable with the incompleteness of Łukasiewicz's earlier 3-valued calculus in C and N as axiomatised by Wajsberg in 1932. An operator "Δp" may be introduced into this system by the definition "$\Delta p = CNpp$", and in this system also $C\Delta pp$ is not disprovable. For the formula CCNppp, which is what $C\Delta pp$ expands to by this definition, is quite consistent with Wajsberg's axioms, and occurs along with them in the ordinary two-valued calculus. It becomes disprovable, however, when the 3-valued system is completed, in the manner of Słupecki, by the addition of the operator "T" (which is not interpretable as a C-N complex) and the axioms CTpNTp and CNTpTp.

2. THE INTERPRETATION OF ŁUKASIEWICZ'S "Δ" AS A VARIABLE FUNCTOR WITH A RESTRICTED RANGE. – The new system of Łukasiewicz has the advantage of completeness over both his own earlier 3-valued system and the systems of Lewis and von Wright; but against this we must put the difficulty of giving his new "Possibly" any intelligible interpretation. Quite

a simple interpretation is suggested, however, by the method he employs in Section 6 to obtain his four-valued matrices. What this suggests is that his operator "Δ" is not to be regarded as a constant functor at all but as a variable one, or rather a partly variable one, with its range of possible values restricted to S and V (instead of ranging, like "δ", over the four values S, V, N and F).

This interpretation will account very simply for the fact, to which Łukasiewicz draws attention, that $\delta\Delta\alpha$ is a thesis if and only if both $\delta S\alpha$ and $\delta V\alpha$ are theses, and for all the other peculiar features of the system. For example, no formulae of the form $\Gamma\alpha$ can be asserted in the system, because "$\Gamma\alpha$" amounts to "$N\Delta N\alpha$", and this is always false for the substitution Δ/V. ("$V\alpha$", and therefore "$VN\alpha$", being always true). And the peculiarities of the functor "∇" arise, on this view, from the fact that this also is a variable functor with a restricted range, in fact with the same range as "Δ", but taking the value S when Δ takes the value V, and *vice versa*. (For "∇p" is defined as "$C\Delta pp$", and $CVpp = Sp$ and $CSpp = Vp$).

Our interpretation is at variance, however, with Łukasiewicz's assertion that "the basic modal logic, and, consequently, every system of modal logic is a many-valued system." But the reason which he gives for this is that "there exists in the two-valued logic no functor of one argument which would satisfy" certain groups of formulae (e.g. $\vdash Cp\Delta p$, $\dashv C\Delta pp$ and $\dashv \Delta p$), and this is surely too sweeping. What ought to have been said is rather that "there exists in the two-valued logic no *constant* functor of one argument which would satisfy, etc." The admission of variable as well as constant propositional functors makes possible a two-valued interpretation both of Łukasiewicz's system and of the Lewis systems. On the interpretation of "Δ" now being suggested, the position is as follows: – A formula such as "$\Delta(2 + 2 = 5)$" has not a third or fourth truth-value, but no truth-value at all, for it is not a proposition but only a propositional function. It only acquires a truth-value when the variable "Δ" has been assigned one of its two possible values, and then it will be either simply true or simply false. "$\Delta(2 + 2 = 5)$" yields a true proposition with the substitution Δ/V and a false one with the substitution Δ/S.

The "basic modal logic" is preserved in this sytem for quite a different reason from that which accounts for its preservation in the other. What this "basic modal logic" amounts to is just this: the form Δp, to be capable of being read as "Possibly p", must be a deductively weaker form than p itself ($\vdash Cp\Delta p$, $\dashv C\Delta pp$), without being so weak as to be tautological ($\dashv\Delta p$); and the rule of extensionality must continue to hold in a system when the form Δp is introduced into it, i.e. formally equivalent formulae must continue to be mutually substitutable in theses. In a system containing no formulae more elementary than δp, these conditions are met by $\Sigma p\delta p$ or $A\delta p\delta Np$, and this (with various axiomatisations) gives us the modalities regulated by the Lewis-von Wright formulae. In the Ł-modal system, Δp contrives to be a deductively weaker form than p without being tautological by standing ambiguously for Sp, i.e. p itself, and for the tautological form Vp.

This interpretation suggests the following alternative axiomatisation of Łukasiewicz's system: Instead of his axioms 2, 3 and 4 we could have the rule that the substitutions Δ/' and /C" (i.e. Δ/S and Δ/V) may be performed in theses, and the axioms $\vdash C\delta pC\delta Cpp\delta\Delta p$ and $\dashv p$ (Łukasiewicz's axiom 1, $\vdash C\delta pC\delta Np\delta q$, being retained). Łukasiewicz's 2, 3 and 4 would then be provable as follows: –

(1) $\vdash C\delta pC\delta Cpp\delta\Delta p$
(2) $\dashv p$
(3) $\vdash Cpp$
(4) $\vdash CpCqp$
(5) $\vdash CpCCpqq$
(5) $p/Cpp, q/p = C(3) - (6)$
(6) $\vdash CCCpppp$
(6) $= C(7) - (2)$
(7) $\dashv CCppp$
(1) $\delta/Cp' = C(3) - C(4)\, q/p - 2$
2. $\vdash Cp\Delta p$
(7) $= 3\,\Delta/C''$
3. $\dashv C\Delta pp$
(2) $= 4\,\Delta/'$
4. $\dashv \Delta p$

In this proof (1) and (2) are our new axioms, and (3), (4) and (5) are derivable from $\vdash C\delta pC\delta Np\delta q$ (they are in fact theses 13, 16 and 32 of Łukasiewicz's paper). With Łukasiewicz's axiomatisation, our (1) and (2) are provable as theorems, and our rule of substitution for "Δ" is derivable metalogically.

3. THE ARISTOTELIAN THESES "CMpMNp" AND "ΣpMp" IN THE DIFFERENT SYSTEMS. – At the beginning of his 1951 paper "On Variable Functors of Propositional Arguments",[4] Łukasiewicz shows that if we combine the Aristotelian modal thesis CMpMNp, "If possibly p then possibly not p", with the law $C\delta pC\delta Np\delta q$, we obtain as a thesis CMpMq, from which the rules for quantifiers enable us to deduce $C\Sigma pMp\Pi qMq$. Aristotle, however, accepted the antecedent of this, ΣpMp, but rejected its consequent, ΠqMq. Agreeing (in this paper) with this acceptance and rejection, Łukasiewicz argues that we must reject the formula CMpMNp. What becomes of these various theses in the Lewis systems and in the new system of Łukasiewicz? The interpretations of these systems here suggested make it a simple matter to work out the answer to this, and to see the reasons behind it.

In Bocheński's *La Logique de Théophraste*, it is pointed out that what lies behind the first Aristotelian thesis – which is given in the stronger form EMp MNp – is the fact that for Aristotle "p is possible" means "Neither p nor not-p is necessary". That is, the "Mp" used in symbolising the Aristotelian thesis is to be regarded not as an alternative for the "Δp" of the modern systems, but as an abbreviation for "$KN\Gamma pN\Gamma Np$". When "Mp" is thus defined, the proof of EMpMNp which is formally set out by Bocheński[5] can be taken over without alteration both in the Lewis systems and in Łukasiewicz's. EMpMNp, and of course the weaker CMpMNp, are therefore asserted in both these modern types of modal system as well as in Aristotle's, provided that

"Mp" be given its Aristotelian meaning. (If it is used in the sense of "Λp", EMpMNp is not a thesis in either type of system). The systems diverge, however, with regard to the other theses $C\delta pC\delta Np\delta q$ and ΣpMp; in Łukasiewicz's system the former is asserted and the latter must be rejected, but ΣpMp is asserted by Lewis and $C\delta pC\delta Np\delta q$ is not asserted in any sense which would make "M" a legitimate substitution for "δ". (Łukasiewicz's proof of CMpMq from $C\delta pC\delta Np\delta q$ and CMpMNp depends upon this substitution).

Łukasiewicz's earlier 3-valued system, with "Δp" defined as "CNpp", is closer at this point to Lewis's systems than to his own later one. For in the three-valued system, $C\delta pC\delta Np\delta q$ is not a law in any sense; and this is not surprising, for the most natural way to take this formula is as a declaration that, whatever p may be, the truth-value of p and the truth-value of Np are the only truth-values there are.[6] And our translations will make it clear why the modal operators of the Lewis calculi are not substitutable for the "δ" of the formula, although those calculi (on our reading of them) are two-valued. If "p", "q", etc. be systematically replaced by "δp", "αp", etc., the formula becomes

(1) $C\delta\delta pC\delta N\delta p\delta\gamma p$,

and both this and the wider formula

(2) $C\epsilon\delta pC\epsilon N\delta p\epsilon\gamma p$

are laws of the C-N-δ-Π system, being derivable by appropriate substitutions in the formula in its original form, $C\delta pC\delta Np\delta q$. But the substitution δ/Δ in this last would amount, on our interpretation of the Lewis-von Wright "Δ", to performing the substitution $\epsilon/\Sigma p$ in formula (2). The result of this would be

(3) $C\Sigma p\delta pC\Sigma pN\delta p\Sigma p\gamma p$.

This is certainly to be rejected (the substitutions δ/' and γ/K'N' will make both antecedents demonstrably true and the consequent demonstrably false); but the variables "δ", "ϵ", etc. are in any case not intended to stand for quantifiers. (The substitution $\epsilon/\Sigma p$ is quite a different matter from the substitution $\delta p/\Sigma p\gamma p$, for which we earlier found a justification). If, then, the substitution δ/Δ represents a substitution of the form $\delta/\Sigma p$ or $\epsilon/\Sigma p$, we may deny its legitimacy without prejudice to the formula of bivalence $C\delta pC\delta Np\delta q$; and similarly if the substitution δ/M, i.e. δ/KNΓ'NΓN', represents a substitution of the form ϵ/KNΠp'NΠpN'.

In the Lewis calculi we are therefore free, and justifiably free, to assert both CMpMNp and ΣpMp. As to the latter formula, our "translation" of $\Sigma pKN\Gamma pN\Sigma Np$ would be $\Sigma\delta KN\Pi p\delta pN\Pi pN\delta p$, and this is provable in the C-N-δ-Π calculus.

In Łukasiewicz's 1953 modal calculus, however, "M" is a legitimate substitution for "δ", and $C\delta pC\delta Np\delta q$ holds, so the proof of CMpMq from CMpMNp which he gives at the beginning of his paper "On Variable Functors" can be incorporated in this system without modification. This means that ⊢CMpMq is a thesis of the system. A consequence of this is that any formula of the form "$M\alpha$" must be rejected as a thesis. For if $M\alpha$ were a thesis, this with CMpMq (p/α, q/p) would give us Mp, i.e. KNΓpNΓp; and this, with CKpqp (p/NΓp, q/NΓNp), would give us NΓp; but this last is a rejected formula in Łukasiewicz's system. If Łukasiewicz's "Δ" is interpreted as a variable two-valued functor with its range restricted to S and V, this generalised rejection $\dashv M\alpha$ is entirely natural; for $M\alpha$, i.e. $KN\Gamma\alpha N\Gamma N\alpha$, is equivalent to $K\Delta N\alpha\Delta\alpha$, and this is always false for the substitution Δ/S. (Thus we have $\dashv M\alpha$ because $K\Delta N\alpha\Delta\alpha$ is contradictory when Δ takes its strongest value, and $\dashv\Gamma\alpha$ because $N\Delta N\alpha$ is contradictory when Δ takes its weakest value). The rejection of the Aristotelian thesis ΣpMp similarly falls into place; for ΣpKΔNpΔp is false for the same substitution. (The simple ΣpΔp is, however, asserted in Łukasiewicz's system, when quantifiers are introduced).

4. ŁUKASIEWICZ AND PARRY. – One other difference between his own system and the Lewis systems which Łukasiewicz does not mention, is worth noting, although we may deal with this point quite formally, without reference to interpretations. It has been shown by Parry[7] that if the reduction formula CΓΔpΓp, which follows from Łukasiewicz's 140 by the substitution p/Np and certain simple transformations, were introduced into the Lewis systems, it would be possible to prove the formula CpΓp, which would make these systems cease to be modal. Parry's proof, however, uses as a thesis the formula ΓCpΔp, which is not a thesis in Łukasiewicz's system, being of the form $\Gamma\alpha$. Łukasiewicz's system is therefore not inconsistent at this point, and in fact within it Parry's disproof of CΓΔpΓp by using ΓCpΔp could be re-stated as a disproof of ΓCpΔp by using CΓΔpΓp. The proofs which Łukasiewicz gives in his appendix do in fact include a disproof of ΓCpp, which is allied to ΓCpΔp, by using CΓNΓpΓNp, which is allied to CΓΔpΓp.[8] Parry's work and Łukasiewicz's alike show that if we assert CpΔp but reject CpΓp, we cannot have both the reduction thesis CΓΔpΓp and the rule $\vdash\alpha \rightarrow \vdash\Gamma\alpha$ (by which $\vdash\Gamma Cp\Delta p$ would be obtainable from $\vdash Cp\Delta p$); Parry accepts a procedure akin to the rule and therefore rejects the reduction thesis, while Łukasiewicz does the opposite. Both courses are consistent; which is to be taken depends on what sort of "possibility" we are using "Δ" to symbolise.

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[1] Jan Łukasiewicz, "A System of Modal Logic", *The Journal of Computing Systems*, Vol. I, No. 3 (1953), pp. 111–149.

[2] In "Three-valued Logic and the Problem of Future Contingents" (*Philosophical Quarterly*, 1953), I have defended Łukasiewicz's view that a three-valued logic is required to express the sense given to "possibility" in Aristotle's *De Interpretatione*, Ch. 9.

[3] That modal expressions may sometimes have the sense of quantifiers, has been noted by Łukasiewicz himself in his *Aristotle's Syllogistic*, par. 5.

[4] Jan Łukasiewicz, "On Variable Functors of Propositional Arguments", Proceedings of the Royal Irish Academy, 54 A 2 (1951).

[5] I. M. Bocheński, *La Logique de Théophraste* (Fribourg 1947), p. 69. (Bocheński uses "S" for Łukasiewicz's "Γ").

[6] Cf. my paper "On Propositions Neither Necessary nor Impossible", *The*

Journal of Symbolic Logic, Vol. XVIII (1953), p. 108.

[7] W. T. Parry, "Modalities in the *Survey* System of Strict Implication", JSL Vol. IV (1939). The relevant portions are the proof of 22.8 on p. 140 and subsequent disproof of 1.9* and 1.91* on pp. 153–4.

[8] See, in[1], the proof of 160 from 140, *via* 141, 142, 143, 159.

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ON THE INTERPRETATION OF A MODAL SYSTEM OF ŁUKASIEWICZ

ALAN ROSS ANDERSON

We shall show that the Ł-modal system of Łukasiewicz[1] is semantically non-complete in the sense of Halldén.[2] We require the following theses of Ł:[3]

(Ł22) $p/Epq, q/CN\delta pN\delta q, r/C\delta q\delta p$ * C(Ł73) $\delta/N\delta'$–C(Ł38) $p/\delta p, q/\delta q$–1

1. $\vdash CEpqC\delta q\delta p$

(Ł72) $p/NNp, q/p$ * C(Ł36)–C(Ł39)–2

2. $\vdash ENNpp$

(Ł43) $p/C\Delta pp, q/\nabla p$ * C(Ł9) $\delta/'$–3

3. $\vdash CN\nabla pNC\Delta pp$

1 $\delta/CN\nabla pNC\Delta p', p/NNp, q/p$ * C2–C3–4

4. $\vdash CN\nabla pNC\Delta pNNp$

(Ł5) $\delta/CN\nabla p', p/\Delta p, q/Np$ * C4–5

5. $\vdash CN\nabla pK\Delta pNp$

(Ł16) $p/CNp\Delta Np, q/K\Delta pNp$ * C(Ł2) p/Np–6

6. $\vdash CK\Delta pNpCNp\Delta Np$

(Ł63) $p/\Delta p, q/Np, r/\Delta Np$ * C6–7

7. $\vdash CK\Delta pNp\Delta Np$

(Ł61) $p/K\Delta pNp, q/\Delta p, r/\Delta Np$ * C(Ł55) $p/\Delta p, q/Np$–C7–8

8. $\vdash CK\Delta pNpK\Delta p\Delta Np$

(Ł22) $p/K\Delta pNp, q/K\Delta p\Delta Np, r/\Delta KpNp$ * C8–C(Ł108) q/Np–9

9. $\vdash CK\Delta pNp\Delta KpNp$

1 $\delta/CNCp'q, p/NNp, q/p$ * C2–C(Ł14)–10

10. $\vdash CNCpNNpq$

(Ł38) $p/NCpNNp, q/KpNp$ * C(Ł5) $\delta/N', q/Np$–11

11. $\vdash CKpNpNCpNNp$

(Ł22) $p/KpNp, q/NCpNNp, r/q$ * C11–C10–12

12. $\vdash CKpNpq$

(Ł2) $p/CKpNpq$ * C12–13

13. $\vdash \Delta CKpNpq$

(Ł77) $p/KpNp$ * C13–14

14. $\vdash C\Delta KpNp\Delta q$

(Ł22) $p/K\Delta pNp, q/\Delta KpNp, r/\Delta q$ * C9–C14–15

15. $\vdash CK\Delta pNp\Delta q$

(Ł22) $p/N\nabla p, q/K\Delta pNp, r/\Delta q$ * C5–C15–16

16. $\vdash CN\nabla p\Delta q$

(Ł6) $\delta/', p/\nabla p, q/\Delta q$ * C16–17

17. $\vdash A\nabla p\Delta q$

(Ł38) $p/C\Delta pp, q/\nabla p$ * C(Ł9) δ/N'–18

18. $\vdash C\nabla pC\Delta pp$

18 * C19–(Ł3)

19. $\dashv \nabla p$

(Ł4) p/q * 20

20. $\dashv \Delta q$

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The semantical non-completeness of Ł follows from 17, 19, and 20, by a theorem of Halldén,[4] which we shall not repeat in detail here. The essential point of Halldén's result as regards the system Ł under consideration can be indicated briefly as follows: $A\nabla p\Delta q$ is an asserted formula of Ł. Suppose all asserted formulae of Ł are true, and that we interpret "A" as "or" in the usual way. Then either ∇p is true (for every p), or Δq is true (for every q).[5] But ∇p and Δq are both rejected. Hence some true formulae of Ł are rejected, and the class of true formulae cannot coincide with the class of asserted formulae. (For details, the reader is referred to Halldén's paper.) Since $\dashv\nabla p$ and $\dashv\Delta q$ are both intuitively plausible, it would appear that some unusual interpretation of "A" is required if we are to preserve semantical completeness.

Thesis 17 also makes questionable the interpretation of both "Δ" and '∇'. Łukasiewicz suggests that they are both interpretable as "possible", though in slightly different senses which are not distinguished in English.[6] However, it is difficult to think of two reasonable senses of "possible" in which, given any two propositions, either the first proposition is possible in the first sense or the second proposition is possible in the second sense. It might be thought that the difficulty here lies in a misinterpretation of '∇'; but even if we replace the latter in 17 by its definition (Ł9), we have 21 $\vdash AC\Delta pp\Delta q$ which seems to shed equal doubt on the interpretation of "Δ".

NOTES

[1] J. Łukasiewicz: A system of modal logic. *The Journal of Computing Systems*, vol. 1 (1952–53), pp. 111–149.

[2] S. Halldén: On the semantic non-completeness of certain Lewis calculi. *The Journal of Symbolic Logic*, vol. 16 (1951), pp. 127–129.

[3] We adopt the notation and proof conventions of Łukasiewicz, op. cit., especially pp. 136–137.

[4] Halldén, op. cit.

[5] It is of course essential that the disjoined formulae in thesis 17 contain no variables in common.

[6] Łukasiewicz, op. cit. pp. 127–130 and 133.

YALE UNIVERSITY

ON ALTERNATIVE FORMULATIONS OF A MODAL SYSTEM OF FEYS-VON WRIGHT

ALAN ROSS ANDERSON

In a recent article[1], B. Sobociński notes that we can formulate a system equivalent to von Wright's system M, which (a) has a finite number of axioms, and (b) dispenses with the first of the following four primitive rules of inference:

RI. If α is a thesis, then $L\alpha$ is a thesis.

RII. If $E\alpha\beta$ is a thesis, then $EM\alpha M\beta$ is a thesis

RIII. If α and $C\alpha\beta$ are theses, then β is a thesis.

RIV. If α is a thesis, and if β results from replacing the propositional variable γ by the well-formed formula δ throughout α, then β is a thesis.

Sobociński's alternative formulation of M results from dropping RI from von Wright's original formulation, and adding as an axiom the thesis LCpp.[2]

On the other hand Feys' system T (which is shown by Sobociński to be equivalent to M) can be formulated with a finite number of axioms, and RI, RIII, and RIV as primitive rules.[3]

The question naturally arises as to whether we can formulate a system R, equivalent to M and T, with the help of RIII and RIV alone as primitive rules of inference. The following formulation (useful in attacking the decision problem[4]) satisfies this condition, though it employs infinitely many axioms; namely:

(1) CpCqp
(2) CCpCqrCCpqCpr
(3) CCNqNpCpq
(4) CpMp
(5) EMApqAMpMq
(6) CLCpqCMpMq
(7) CLCpqCLpLq

together with any axiom generated from (1)–(7) by the rule: if α is an axiom, then $L\alpha$ is an axiom.

To prove that R is equivalent to M, we need only show that RI and RII are derivable for R, since the remainder of the proof is trivial. We can verify that RI is a rule of R by induction on the length of the proof of α, using (7) and the rule for generating axioms. Then prove RII for R, with the help of RI and (6). Then R can easily be shown equivalent to M in the usual way.

Systems equivalent to S4 (S5) arise by adding the thesis CMMpMp (the thesis CMNMpNMp) to the axioms of R. We notice that CLpp is provable from axioms (1)–(4) with the aid of RIII and RIV, and without using any axioms of the form $L\alpha$. Hence in these axiomatizations of R (i.e., M), S4, and S5, we can omit any finite number of axioms of the form $L\alpha$ without affecting the strength of the respective systems; hence each axiomatization contains any finite number of dependent axioms. In fact, if we omit e.g. all axioms of the form L_nCpMp, where L_n is a string of nL's, and n is even and greater than zero, we still leave each system unaffected as to strength; hence each formulation contains an infinite number of dependent axioms.

It is known that these infinite sets of axioms for S4 and S5 can be replaced

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by equivalent finite sets of axioms, still keeping RIII and RIV as the only primitive rules[5]; but the analogous question for R remains open. The writer conjectures that no such replacement is possible for R.

NOTES

1) B. Sobociński: Note on a modal system of Feys-von Wright. *The Journal of Computing Systems*, vol. 1 (1952–53), pp. 171–178. We adopt Sobociński's notation, and presuppose familiarity with his article and the literature there cited. We take "C", "N", and "M", as undefined, and assume the other constants to be defined appropriately. The author is indebted to Dr. Sobociński for a correction to an earlier draft of this note.

2) Sobociński op. cit., 4.2.

3) Sobociński op. cit., 1.3. and 3.

4) See the author's forthcoming paper "*Improved decision procedures for Lewis's calculus S4 and von Wright's calculus M*", an abstract of which appears in the *Journal of Symbolic Logic*, vol. 18 (1953), pp. 187–188.

5) The twelve axiom schemata of McKinsey-Tarski$_1$(cited by Sobociński op. cit.) may easily be replaced by RIV and a finite number of proper axioms.

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ARITHMETIC AND MODAL LOGIC

JAN ŁUKASIEWICZ

1. It is well known that the theory of equality can be based on the following two axioms:

1.1 $\vdash \Theta aa$, i.e. "a is equal to a",

1.2 $\vdash C\Theta abC\phi a\phi b$, i.e. "If a is equal to b, then if ϕ a, ϕ b."

ϕ is a variable proposition-forming functor of one numerical argument, and we are allowed to substitute for ϕ any expression which gives together with the argument of ϕ a significant arithmetical formula.[1]

If we introduce into arithmetic the modal functor Γ ("it is necessary that") or Δ ("it is possible that"), then expressions as: $\Gamma\Theta ab$, i.e. "It is necessary that a should be equal to b", or $\Delta N\Theta ab$, i.e. "It is possible that a is not equal to b", will be significant arithmetical formulae. We have therefore:

$$1.2.\ \phi/\Gamma\Theta a' \ * \ 1.3$$

1.3 $\vdash C\Theta abC\Gamma\Theta aa\Gamma\Theta ab$,

in words: "If a is equal to b, then if it is necessary that a should be equal to a, it is necessary that a should be equal to b". From 1.3 we get by the law of commutation

1.4 $\vdash CCpCqrCqCpr$

the formula 1.5:

$$1.4\ p/\Theta ab, q/\Gamma\Theta aa, r/\Gamma\Theta ab \ * \ C1.3-1.5$$

1.5 $\vdash C\Gamma\Theta aaC\Theta ab\Gamma\Theta ab$.

Most logicians would assert the antecedent of 1.5:[2]

(a) $\Gamma\Theta aa$,

in words: "It is necessary that a should be equal to a." Hence they must accept the consequent:

(b) $C\Theta ab\Gamma\Theta ab$.

That means: Equality holds necessarily if it holds at all.

From (b) there follows by means of the modal theorem

1.6 $\vdash C\Gamma pN\Delta Np$,

and the principle of the syllogism

1.7 $\vdash CCpqCCqrCpr$

the consequence (c):

$$1.7\ p/\Theta ab, q/\Gamma\Theta ab, r/N\Delta N\Theta ab \ * \ C(b)-C1.6p/\Theta ab-(c)$$

(c) $C\Theta abN\Delta N\Theta ab$,

and from (c) we get by the law of transposition

1.8 $\vdash CCpNqCqNp$

the formula (d):

$$1.8\ p/\Theta ab, q/\Delta N\Theta ab \ * \ C(c)-(d)$$

(d) $C\Delta N\Theta abN\Theta ab$.

That means: "If it is (only) possible that a is not equal to b, then a is (factually) not equal to b."

2. I have not prefixed the sign of assertion "$\vdash$" to the formulae (a), (b), (c), and (d), as these formulae are in my opinion wrong and should be rejected. In particular formulae (b) and (d) are obviously false. Quine gives an example for the falsity of (b): Let a denote "the number of planets", and b the number

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''9''. It is afactual truth that the number of (major) planets is equal to 9, but it is by no means necessary that it should be equal to 9. Quine tries to meet this difficulty by raising objections to the substitution of such singular terms for the variables. ''Such instantiation – he writes – is allowable, certainly, in extensional logic; but it is a question of good behaviour of constant singular terms, and ... such behaviour is not to be counted on when there is a 'nec' in the woodpile.''[2] Quine does not explain, however, how this ''naughty'' behaviour of singular terms should be corrected. His remark is a desideratum rather than a solution of the problem.

For the falsity of formula (d), not mentioned by Quine, take the following example: Let us suppose that the number a has been thrown with a die. It is possible that the number b next thrown with the die will be different from a. But if it is only possible that a will be different from b, i.e. not equal to b, then according to (d) a will factually be different from b. This consequence is obviously wrong, as it is possible to throw the same number twice.

3. The Ł–modal system expounded by myself in this Journal[3] gives a satisfactory solution of the above difficulties. This system is based on two asserted axioms:

$3.1 \vdash C\delta pC\delta Np\,\delta q$

$3.2 \vdash Cp\Delta p.$

The first axiom which means: ''If p satisfies the condition δ, then if Np satisfies δ, any proposition q satisfies δ'', yields the whole two-valued classical calculus of propositions and all the asserted δ–formulae. The second is a well-known modal theorem which gives together with the axiom 3.1 a complete system of modal logic. Both axioms are perfectly evident. Two axioms of rejection (''$\dashv$'' is the sign of rejection):

$3.3 \dashv C\Delta pp$

$3.4 \dashv \Delta p$

are needed to characterize the system as a modal logic.

The rules of inference are the rule of substitution and detachment for the asserted and rejected formulae.

The system has the following adequate four-valued matrix M1:

C	1	2	3	4	N	Δ	∇	Γ
*1	1	2	3	4	4	1	1	2
2	1	1	3	3	3	1	2	2
3	1	2	1	2	2	3	1	4
4	1	1	1	1	1	3	2	4

M1

Γp is defined by $N\Delta Np$, ∇ by $C\Delta pp$.

A consequence of the system is the asserted formula:

$3.5 \vdash C\Gamma qCp\Gamma p.$

The consequent of this formula is rejected:

$3.6 \dashv Cp\Gamma p.$

We have therefore for any proposition α:

$3.5\ q/\alpha\ *\ C3.7\text{–}3.6$

$3.7 \dashv \Gamma\alpha.$

That means that no apodeictic proposition can be asserted in the Ł-modal system. Asserted propositions are merely true without being necessary. From 3.7 we get by interpreting a as Θaa:

$3.8 \dashv \Gamma\Theta aa$,

i.e. formula (a) is rejected. From

$3.9 \vdash CpCCpqq$ and .

$3.10 \vdash \Theta aa$

there follows the consequence 3.11:

$3.9\ p/\Theta aa, q/\Gamma\Theta aa * C3.10 - 3.11$

$3.11 \vdash CC\Theta aa\Gamma\Theta aa\Gamma\Theta aa.$

Applying to 3.11 the rule of detachment for rejected formulae we get 3.12:

$3.11 * C3.12 - 3.8$

$3.12 \dashv C\Theta aa\Gamma\Theta aa,$

and by the rule of substitution for rejected formulae we have:

$3.12 * 3.13\ b/a$

$3.13 \dashv C\Theta ab\Gamma\Theta ab.$

That means that formula (b) is rejected. In a similar way we can prove that formulae (c) and (d) must be rejected too. Here are the respective deductions:

$3.14 \vdash CN\Delta Np\Gamma p$ (follows from the definition of Γp)

$3.15 \vdash CCqrCCpqCpr$

$3.15\ q/N\Delta N\Theta ab, r/\Gamma\Theta ab, p/\Theta ab * C3.14 p/\Theta ab - 3.16$

$3.16 \vdash CC\Theta abN\Delta N\Theta abC\Theta ab\Gamma\Theta ab$

$3.16 * C3.17 - 3.13$

$3.17 \dashv C\Theta abN\Delta N\Theta ab,$ i.e., formula (c) is rejected.

$3.18 \vdash CCpNqCqNp$

$3.18\ p/\Delta N\Theta ab, q/\Theta ab * 3.19$

$3.19 \vdash CC\Delta N\Theta abN\Theta abC\Theta abN\Delta N\Theta ab$

$3.19 * C3.20 - 3.17$

$3.20 \dashv C\Delta N\Theta abN\Theta ab,$ i.e., formula (d) is rejected.

On the other side, the asserted formula $C\Theta abC\Gamma\Theta aa\Gamma\Theta ab$, which is correctly deduced from the axiom $C\Theta abC\phi a\phi b$, is easily verified by the matrix M1. As Θaa is asserted and has the value 1, $\Gamma\Theta aa = \Gamma 1 = 2$. We have therefore to verify the formula

$C\Theta abC2\Gamma\Theta ab$

for all possible values of Θab. We get:

For $\Theta ab = 1$: $C1C2\Gamma 1 = C1C22 = C11 = 1$;
" $\Theta ab = 2$: $C2C2\Gamma 2 = C2C22 = C21 = 1$;
" $\Theta ab = 3$: $C3C2\Gamma 3 = C3C24 = C33 = 1$;
" $\Theta ab = 4$: $C4C2\Gamma 4 = C4C24 = C43 = 1$.

4. Propositions, as "a is equal to a", are called "analytic". The well-known doctrine that all analytic propositions are necessary goes back to Aristotle who distinguishes between essential and accidental properties and asserts that essential properties belong to the things with necessity.[4]

Essential properties are based on definitions, i.e. on the meaning of words. So for instance, "Man is necessarily an animal"[5], because "man" is defined as an "animal". In view of the formulae $\vdash C\Gamma pp$ and $\dashv Cp\Gamma p$ it is commonly held that apodeictic propositions have a higher dignity and are more reliable than corresponding assertoric ones. This consequence is for me by no means evident. I cannot understand why the true proposition based on the meaning of words "I am an animal" should be more reliable than the factual truth based on experience "I have brown eyes". Another Aristotle's argument connected with the subject and sometimes called "the Aristotelian paradox" is still less evident. Aristotle asserts: "If it is true to say that something is white or not white, it is necessary that it should be white or not white."[6] As it is impossible to translate this statement by the false formula $Cp\Gamma p$, some logicians accept as a rule that it is allowed to infer from an asserted (analytic) proposition α the asserted apodeictic proposition $\Gamma\alpha$. This agains leads to asserted apodeictic propositions, and if we accept such propositions at all, we are bound to assert the necessity of the principle "a is equal to a". In view of the difficulties which would result from this fact, I am inclined to think that all systems of modal logic which accept asserted apodeictic propositions are wrong.

A system of this kind is my three-valued modal logic constructed by the matrix M2 in 1920, and developed in 1930.[7]

$$\begin{array}{c|ccc|c}
C & 1 & 2 & 3 & N \\
\hline
{}^{*}1 & 1 & 2 & 3 & 3 \\
2 & 1 & 1 & 2 & 2 \\
3 & 1 & 1 & 1 & 1
\end{array}$$

M2

C is here a kind of implication different from the ordinary one, and N a kind of negation. The system for C and N was axiomatized by M. Wajsberg[8], and extended to a complete system by the addition of a new function by J. Słupecki[9]. In my paper of 1930 I accepted the definition of possibility suggested by A. Tarski:

(t) $\Delta p = CNpp$

which is equivalent to the definition of necessity:

(u) $\Gamma p = NCpNp$.

It is easy to see on the ground of the matrix M2 that in this system the formula

(v) $CpCp\Gamma p = CpCpNCpNp$

is verified, and yields not only ΓCpp for p/Cpp, but also the Aristotelian paradoxical rule. I think therefore that my three-valued modal logic cannot be regarded as an adequate system of modal logic.

The same remark should be made about the Lewis's systems of "strict implication". The function "p strictly implies q" is defined by Lewis by the expression "It is not possible that p and not q"[10]. From this definition we can easily deduce that the formula "p strictly implies q" is an asserted apodeictic proposition. The systems of Lewis are certainly very interesting and may have their own merits; I think, however, that they cannot be regarded as adequate systems of modal logic.

5. Modal logic is important as a theory of possibility. There exist true problematic propositions which would not be true as assertoric propositions. There are other true propositions which cannot be proved without introducing

possibility. Both kinds of proposition extend our knowledge beyond the stock of truths which can be got by the non-modal logic.

I shall explain here an application of my Ł-modal system to arithmetic which throws a singular light on the meaning of the so called "existential quantifier". From the principle of identity

5.1 $\vdash Cpp$

we get by substitution

5.1 $p/\phi a$ * 5.2

5.2 $\vdash C\phi a\phi a$,

and by the rule of quantifiers denoted by $\Sigma 2$ the formula 5.3:

5.2 $\Sigma 2a$ * 5.3

5.3 $\vdash C\phi a\Sigma a\phi a$.

In words: "If ϕ of a, then for some a ϕ of a", or "If ϕ of a, then there exists such an a that ϕ of a". "a" denotes any positive integer, i.e. any number of the sequence 1,2,3,..., in inf.

By means of the functors Δ and ∇ (see above sect. 3) it is possible to construct true expressions of the form $\Sigma a\phi a$, though there exists no positive integer a that would verify ϕa. The simplest expression of this kind is the following one:

5.4 $\vdash \Sigma aK\Delta\Theta 1a\nabla L1a$,

where "$L1a$" means "1 is less than a". For a = 1, $\Theta 1a$ is true and $L1a$ is false, or $\Theta 1a = 1$ and $L1a = 4$; for $a > 1$, $\Theta 1a$ is false and $L1a$ is true, or $\Theta 1a = 4$ and $L1a = 1$. We have therefore:

If $a = 1$, then $K\Delta\Theta 1a\nabla L1a = K\Delta 1\nabla 4 = K12$,
if $a > 1$, then $K\Delta\Theta 1a\nabla L1a = K\Delta 1\nabla 1 = K31$.

Now according to the matrix for $Kpq = NCpNq$:

K	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

the conjunction $K12 = 2$, and $K31 = 3$, i.e. none of them is equal to 1. Hence it appears that no positive integer verifies the conjunction $K\Delta\Theta 1a\nabla L1a$. Nevertheless the quantified expression $\Sigma aK\Delta\Theta 1a\nabla L1a$ is true. I give here a full proof of this theorem based on asserted formulae of the Ł-modal system and on three elementary arithmetical theses.

The premises.

1 $\vdash Cpp$ (14)
2 $\vdash CCpqCCqrCpr$ (17,22,23)
3 $\vdash CNpCpq$ (20)
4 $\vdash CpCqKpq$ (16)

5 $\vdash CpCqKqp$ (19)
6 $\vdash CCprCCqrCApqr$ (24)
7 $\vdash Cp\Delta p$ (15)
8 $\vdash Cp\overline{\nabla}p$ (18)
9 $\vdash CCpqC\Delta p\Delta q$ (21)
10 $\vdash A\Delta p\nabla p$ (24)
11 $\vdash \Theta aa$ (15) $(a = a)$
12 $\vdash NLaa$ (20) $(a<a)$
13 $\vdash La\Xi a1$ (18) $(a<a+1)$

The deduction.

1 $p/K\Delta\Theta 1a\nabla L1a, \Sigma 2a * 14$
14 $\vdash CK\Delta\Theta 1a\nabla L1a\Sigma aK\Delta\Theta 1a\nabla L1a$ (17,23)
7 $p/\Theta 11 * C11a/1 - 15$
15 $\vdash \Delta\Theta 11$ (16)
4 $p/\Delta\Theta 11, q/\nabla L11 * C15 - 16$
16 $\vdash C\nabla L11K\Delta\Theta 11\nabla L11$ (17)
2 $p/\nabla L11, q/K\Delta\Theta 11\nabla L11, r/\Sigma aK\Delta\Theta 1a\nabla L1a * C16 - C14a/1 - 17$
17 $\vdash C\nabla L11\Sigma aK\Delta\Theta 1a\nabla L1a$ (24)
8 $p/L1\Xi 11 * C13a/1 - 18$
18 $\vdash \nabla L1\Xi 11$ (19)
5 $p/\nabla L1\Xi 11, q/\Delta\Theta 1\Xi 11 * C18 - 19$
19 $\vdash C\Delta\Theta 1\Xi 11K\Delta\Theta 1\Xi 11\nabla L1\Xi 11$ (22)
3 $p/L11, q/\Theta 1\Xi 11 * C12a/1 - 20$
20 $\vdash CL11\Theta 1\Xi 11$ (21)
9 $p/L11, q/\Theta 1\Xi 11 * C20 - 21$
21 $\vdash C\Delta L11\Delta\Theta 1\Xi 11$ (22)
2 $p/\Delta L11, q/\Delta\Theta 1\Xi 11, r/K\Delta\Theta 1\Xi 11\nabla L1\Xi 11 * C21 - C19 - 22$
22 $\vdash C\Delta L11K\Delta\Theta 1\Xi 11\nabla L1\Xi 11$ (23)
2 $p/\Delta L11, q/K\Delta\Theta 1\Xi 11\nabla L1\Xi 11, r/\Sigma aK\Delta\Theta 1a\nabla L1a * C22 - C14a/\Xi 11 - 23$
23 $\vdash C\Delta L11\Sigma aK\Delta\Theta 1a\nabla L1a$ (24)
6 $p/\Delta L11, q/\nabla L11, r/\Sigma aK\Delta\Theta 1a\nabla L1a * C23 - C17 - C10p/L11 - 24$
24 $\vdash \Sigma aK\Delta\Theta 1a\nabla L1a$

It follows from this consideration that it would be wrong to translate the expression Σa by the phrase "for some a" or "there exists such an a that". In order to express in words a formula of the shape $\Sigma a\phi a$, we must first transform it into the equivalent form $N\Pi aN\phi a$, and then say accordingly: "It is not the case that for all a not ϕ of a." It seems to me that the philosophical implications of this logical fact may be of some importance.

Notes

1. For the explanation of my symbolic notation in arithmetic see: J. Łukasiewicz, "Sur la formalisation des théories mathematiques", *Colloques internationaux du Centre national de la recherche scientifique*, XXXVI *Les méthodes formelles en axiomatique*, Paris 1953, pp. 11–19.

2. W. V. Quine, "Three Grades of Modal Involvement", *Proceedings of the XIth International Congress of Philosophy*, Vol. XIV, Brussels 1953, p. 80: "... surely 'nec(x = x)' is true for all x." – 'nec' means in our notation "Γ".

3. J. Łukasiewicz, "A System of Modal Logic", *The Journal of Computing Systems*, Vol. I, No. 3, St. Paul 1953, pp. 111–149.

4. An. post. A 6, 74 b 6: τὰ δὲ καθ᾿ αὑτα ὑπάρχοντα ἀναγκαῖα τοῖς πράγμασιν.

5. An. pr. A 9, 30 a 30: ζῷον μὲν γὰρ ὁ ἄνθρωπος ἐξ ἀνάγκης ἐστί.

6. De int. 9,18 a 39: εἰ γὰρ ἀληθὲς εἰπεῖν ὅτι λευκὸν ἢ ὅτι οὐλευκόν ἐστιν, ἀνάγκη εἶναι λευκὸν ἢ οὐ λευκόν.

7. J. Łukasiewicz, "O logice trójwartościowej", *Ruch Filozoficzny*, V, Lwów 1920. – J. Łukasiewicz, "Philosophische Bermerkungen zu mehrwertigen Systemen des Aussagenkalküls", *Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie*, Vol. XXIII, Cl. 3, 1930.

8. M. Wajsberg, "Aksjomatyzacja trójwartościowego rachunku zdań", *Comptes Rendes des séances de la Société des Sciences et des Lettres de Varsovie*, Vol. XXIV, Cl. 3, 1931.

9. J. Słupecki, "Pełny trójwartościowy rachunek zdań", *Annales Universitatis Mariae Curie–Skłodowska*, Vol. I, Nr 3, Sectio F, Lublin 1946.

10. C. I. Lewis and C. H. Langford, "Symbolic Logic", New York and London (1932), p. 124.

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NOTE ON THE COMPLETENESS OF DECISION ELEMENT SETS

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In a previous article,[1] the author indicated the complete sets of decision elements. The result there announced holds provided a delay element is available. A one unit delay can be synthesized from any one of the following elements: A, B, C, E, G, H, K, L, R, and T. It can be readily seen that none of the other DE's will do as induction will show that, expressing the output f as a function of the inputs, if there is an odd number of delays and f is synthesized from D, S, J, P, F and V exclusively, then either $f(1, 1, \ldots, 1) = f(0, 0, \ldots, 0)$ or $f(1, 1, \ldots, 1) = 0$ and $f(0, 0, \ldots, 0) = 1$.[2] Accordingly, sets 3–14 are complete and 1 or 2 with one of A, B, C, E, G, H, K, L, R and T is also complete. On the basis of this result, in the present stronger sense of completeness (i.e., completeness without the assumption of a delay element), we have the following minimal sets: 3–14 of the previous paper, and one of D and S plus one of A, B, E, G, K, and R.

NOTES

1. Norman M. Martin, "On Completeness of Decision Element Sets", *Journal of Computing Systems,* Vol. I, No. 3, (July, 1953), pp. 150–154.

2. It is of interest to note that a consequence of this result is that contrary to common impression, in the strong sense of completeness of this note, neither D nor S are complete despite their completeness in truth-function theory, while on the other hand, as indicated previously, if the clock pulse is available or if one of the states is represented by the absence of a pulse, other DE's, not complete in truth-function theory are complete. This point has also been raised, independently, by Goodell, "Notes on Decision Element Techniques using Various Practical Techniques", *Journal of Computing Systems,* Vol. I, No. 3, (July, 1953), pp. 197–198.

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EXPRESSIONS TRIVIALLY DECIDABLE

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1. It can be shown that certain expressions of the propositional calculus involving a large number of variables (n) are trivially decidable in a fashion to be outlined, thereby avoiding the necessity of submitting such expressions for a machine-decision which, where, say, $n > 25$, may require a considerable completion-time (t).

2. Where 'o' = 'falsehood' and '1' = 'truth', and the truth-matrix is of the type:

$$\begin{array}{llllll} a: & o\,1\,o\,1 & o\,1\,o\,1 & o\,1\,o\,1 & o\,1\,o\,1 & \ldots \\ b: & o\,o\,1\,1 & o\,o\,1\,1 & o\,o\,1\,1 & o\,o\,1\,1 & \ldots \\ c: & o\,o\,o\,o & 1\,1\,1\,1 & o\,o\,o\,o & 1\,1\,1\,1 & \ldots \\ \cdot & \cdot\,\cdot\,\cdot\,\cdot & \cdot\,\cdot\,\cdot\,\cdot & \cdot\,\cdot\,\cdot\,\cdot & \cdot\,\cdot\,\cdot\,\cdot & \ldots \end{array} \qquad (1)$$

then the patterns of 2^n digits in each row may for computation-purposes, be represented by the use of 2^p digits, (where $p < n$), as instructions for linear distribution of unit-patterns of the same order: thus with reference to the 'a' and 'b' rows of (1):

$$a = \begin{vmatrix} o & 1 & o & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} \qquad b = \begin{vmatrix} o & o & 1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} \qquad (2)$$

(i) (ii)

i.e. the instructions '1111' directs the four-fold repetition of the unit 'o1o1' in (i) and of 'oo11' in (ii). The 'c'-row may be represented in two ways:

$$c = \begin{vmatrix} o & o & o & o \\ 1 & o & 1 & o \end{vmatrix}^{1} \qquad c = \begin{vmatrix} 1 & 1 & 1 & 1 \\ o & 1 & o & 1 \end{vmatrix}^{o} \qquad (3)$$

(i) (ii)

Here the instruction contains o-digits. If these are ignored, (i) becomes oooo.... oooo...., and (ii) becomes ....1111....1111. The context-indices (1 and o respectively) direct the manner in which the blanks corresponding to the o's of the instruction-patterns are to be completed, i.e. with 1's (for the 1-index), and with o's (for the o-index). Forms such as (i) and (ii) of (2) and (3) may be termed logarithms, and may be indefinitely extended, further instructions being added as to the distribution of already-distributed units. Thus the extensions of the matrix-rows of (1) may be expressed logarithmically:

o-*context*:

n	a	b	c	d	e	f	g	h
2	o1o1	oo11	1111	1111	1111	1111	1111	1111
4	1111	1111	o1o1	oo11	1111	1111	1111	1111
6	1111	1111	1111	1111	o1o1	oo11	1111	1111
8	1111	1111	1111	1111	1111	1111	o1o1	oo11

(4)

etc.

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1-*context*

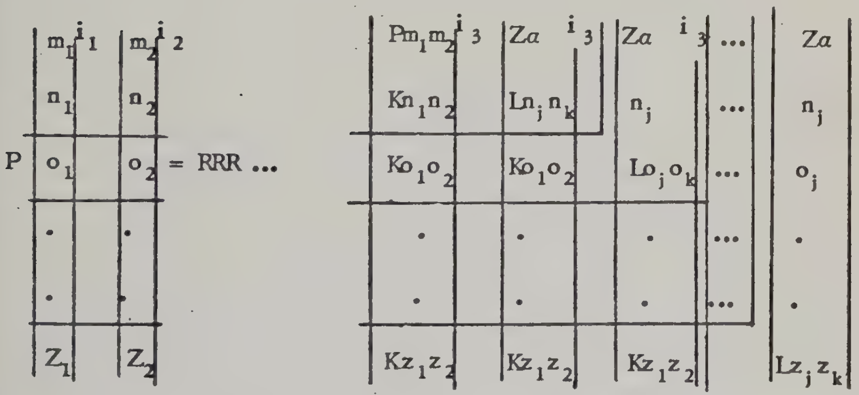
n	a	b	c	d	e	f	g	h
2	0101	0011	0000	0000	0000	0000	0000	0000
4	1111	1111	1010	1100	1111	1111	1111	1111
6	1111	1111	1111	1111	1010	1100	1111	1111
8	1111	1111	1111	1111	1111	1111	1010	1100

etc.

(5)

In general, the truth-table, based on (1), of a function of n variables may be depicted either as a single logarithm (where one unit is uniformly distributed), or as the sum of several 0-context logarithms, or as the product of several 1-context logarithms; (see para. 3).

3. Now where i is the context-index, and P and R represent the operators mentioned in table (7), the outcome of operations performed on single logarithms may be computed by means of the following expression, diversified in accordance with that table. The type of stage to be added for each instruction-increment is indicated by dividing-lines:



(A) (B) (C)

(6)

(7)

No.	P	i_1	i_2	i_3	R	Z_a $j = 1$ $k = 2$	Z_b $j = 2$ $k = 1$	No.	P	i_1	i_2	i_3	R	Z_a $j = 1$ $k = 2$	Z_b $j = 2$ $k = 1$
1	A	1	1	1				15	A	0	0	0	A	m_1	m_2
2	K	0	0	0				16	K	1	1	1	K	m_1	m_2
3	S	1	1	0				17	S	0	0	1	K	Nm_1	Nm_2
4	D	0	0	1				18	D	1	1	0	A	Nm_1	Nm_2
5	L	0	1	0				19	C	1	0	0	A	Nm_1	m_2
6	C	0	1	1				20	L	1	0	1	K	m_1	Nm_2
7	K	0	1	0	A	m_1		21	E	0	0	1	K	Nm_1	Nm_2
8	D	0	1	1	K	Nm_1		22	E	0	1	0	A	m_1	Nm_2
9	C	0	0	1	K	Nm_1		23	E	1	1	1	K	m_1	m_2
10	L	0	0	0	A	m_1		24	J	0	0	0	A	m_1	m_2
11	C	1	1	1	K		m_2	25	J	0	1	1	K	Nm_1	m_2
12	L	1	1	0	A		Nm_2	26	J	1	1	0	A	Nm_1	Nm_2
13	A	0	1	1	K		m_2								
14	S	0	1	0	A		Nm_2								

The right-hand side of (6) is the form of a complex logarithm whose distribution instructions are such that the units distributed are non-interfering, i.e. no two sets of instructions site their unit in the same position. Where no Z_a-value appears in the table, the corresponding outcome logarithm is not required for the context/operator combinations in question. Expression (6) in itself thus forms the first part of a type of decision proceedure which avoids the usual 2^n rate of increase in the number of operations required at each step. Expressions for operations on complex logarithms also exist. These and (6) may be proved by ordinary decision methods, provided that a matrix of the type shown at (1) is in use, since (i) where i = o the logarithm is then the equivalent of the product of m(a, b), n(c,d), o(e,f) etc., where m, n and o are the operators defined by the truth-tables set up by m, n, o individually; (ii) where i = 1 the logarithm is the equivalent of the sum of m(a,b), Nn(c,d), No(e,f) etc.

4. From table (7) it is evident that form (A) of (6) alone is needed to determine the logarithmic truth-tables of those formulae which, when expressed in terms of operators (1) to (6), (amongst which are the universal decision elements S and D), would at no step, mediate or ultimate, involve recourse to context-combinations other than those correlated to such operators. It is in this sense that such expressions may be said to be trivially decidable. In determining whether such decidability obtains, advantage may be taken of the two alternative contexts which may be allocated to variables at the outset, (see (4) and (5)).

5. Example:

CKSACa,b,ADcdAefCghLKSijSklCSmnCopCSACqrstCSKuvCwxDSyzKa_2b_2, (8)

Here, since n = 28, and there are 27 steps, the total number of operations required for a mechanical solution of this expression would normally be 27×2^{28}; thus, at 1 operation per second, t = 47 years. Substituting the context-combinations permitted by 1–6 of table (7) produces:

CKSACo1ADooA11Co1LKS11CS11Co1CSACo111CSKooCo1DS11Koo
= CKSA1A111LKooCo1CSA111CSo1Doo
= CKSA111Lo1CS11Co1 (9)
= CKS11oCo1
= CKoo1
= Co1
= 1

(8) is hence decidable solely by means of (A) of (6). In fact when the unit and instructions each consist of a group of four binary digits, and each group is abbreviated by decimalising its three right-hand digits and adding a dot only if the left-hand digit is a 1, then the logarithmic form of the truth-table of expression (8), based on the matrix (1), becomes:

$$\left| \dot{7}\,1\,\dot{0}\,4\,\dot{0}\,\dot{0}\,\dot{0}\,4\,4\,\dot{0}\,1\,4\,\dot{0}\,1 \right|^{1} \qquad (10)$$

In this horizontal statement of the evaluation, the distributed unit is placed on the left, and the final instruction on the right. 14 × 27, i.e. 378 operations only are required to arrive at this result when abbreviations of the form described, or else sextenary numbers, are employed with the appropriate g × g tables; in unabbreviated binary terms, 1512 are necessary.

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ON A NINE-VALUED PROPOSITIONAL CALCULUS

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We show that the nine-valued logic which is represented by figure 1 is not Kuroda's logic but is weakly Brouwerian one.

1. Definitions. In classical logic the following laws hold:

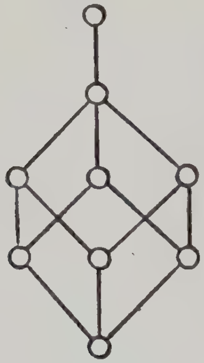
L0. the distributive lattice with 0 and I
L1. $p \cap p^* = 0$
L2. $p \cup p^* = I$
L3. $p = p^{**}$
L4. $(p \cup q)^* = p^* \cap q^*$
L5. $(p \cap q)^* = p^* \cup q^*$
L5′. $(p \cup q)^* = (p^* \cap q^*)^{**}$

where * is negation and L5′ is weaker than L5.

We call the logics which satisfy L0, L1, L4, L5 or L0, L1, L4, L5′ or L0, L1, L4, respectively De Morgan's or Kuroda's or weakly Brouwerian logic. Moreover, in the last logic the next two conditions

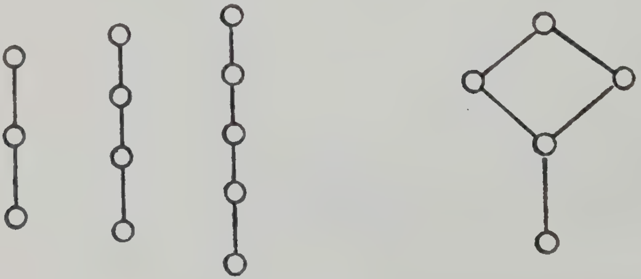
L3$_{\circ}$ $p \rightarrow p^{**}$
L5$_{\circ}$ $p^* \quad q^* \rightarrow (p \quad q)^*$

hold.



(fig. 1)

2. Lattice representations. The examples of lattice representations of De Morgan's logic are known as follows*:



(fig. 2)

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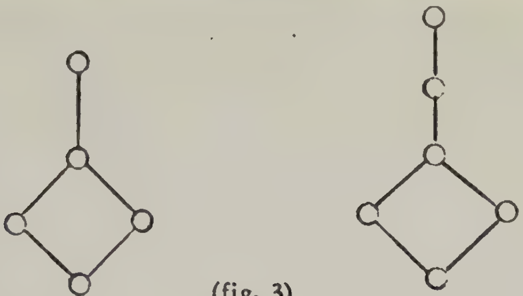
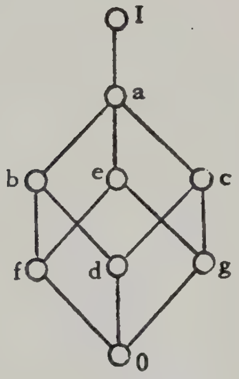
The examples of lattice representations of Kuroda's logic are known as follows*:



(fig. 3)

We prove here that the nine-valued logic whose lattice representation is given by the figure 4 is not Kuroda's logic but is weakly Brouwerian one.



(fig. 4)

At first it is clear that the lattice satisfies the condition L0, for relative complements in it are uniquely determined.

Secondly let "$p \cup q$" denote the proposition that either p or q or both, and let "$p \cap q$" denote the proposition that both p and q, and we define implication and negation by the following table.

---

* T. Sugihara, Negation in many-valued logic, *Memoirs of Liberal Arts College,* Fukui University, vol. 1 (1952) pp. 1–5.

(q)

	$p \to q$	I	a	b	c	d	e	f	g	0	p^*	p^{**}
	I	I	a	b	c	d	e	f	g	0	0	I
	a	I	I	b	c	d	e	f	g	0	0	I
	b	I	I	I	c	d	e	f	g	0	0	I
	c	I	I	I	I	d	e	f	g	0	0	I
(p)	d	I	I	I	I	I	e	f	g	0	0	I
	e	I	I	I	I	I	I	f	g	0	0	I
	f	I	I	I	I	I	I	I	g	0	g	f
	g	I	I	I	I	I	I	I	I	0	f	g
	0	I	I	I	I	I	I	I	I	I	I	0

(Table 1)

Then,

L1 $p \cap q^* = 0$ holds clearly by the table.

L2 $p \cup p^* = I$ does not hold, for $p \cup p^* \to I$ holds but $I \to p \cup p^*$ does not hold. For example:

(1) when $p = a$, $p \cup p^* = a \cup 0 = a$, hence $(I \to p \cup p^*) = (I \to a) = a$.

(2) when $p = f$, $p \cup p^* = f \cup g = e$, hence $(I \to e) = e$.

L3 $p = p^*$ does not hold, for $L3^{\circ}$ $p \to p^{**}$ holds but $p^{**} \to p$ does not hold. For example when $p = a$, $(p^{**} \to p) = (I \to a) = a$.

L4 $(p \cup q)^* = p^* \cap q^*$ holds always.

L5 $(p \cap q)^* = p^* \cup q^*$ does not hold, for $L5^{\circ}$ $p^* \cup q^* \to (p \cap q)^*$ holds but $(p \cap q)^* \to p^* \cup q^*$ does not hold. The examples are written in the table 2.

L5′ $(p \cap q)^* = (p^* \cup q^*)^{**}$ does not hold for $(p^* \cup q^*)^{**} \to (p \cap q)^*$ holds but $(p \cap q)^* \to (p^* \cup q^*)^{**}$ does not hold. The examples are written in the table 2.

p	q	$(p \wedge q)^*$	$p^* \vee q^*$	$(p^* \vee q^*)^{**}$
b	e	g	0	0
b	g	I	f	f
c	e	f	0	0
c	f	I	g	g
d	e	I	0	0
d	f	I	g	g
d	g	I	f	f
f	g	I	e	I

(Table 2)

Thus it is proved that the nine-valued logic of the figure 4 is not Kuroda's logic but is weakly Brouwerian one.

Moreover, it serves to show the independence of L5′ from L0, L1, L4.

3. The open question.

Is the system (figure 4) the smallest (in number of values) which satisfies L0, L1, L4 but not L5?

This is the open question.

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AXIOMATIZATION OF A CONJUNCTIVE-NEGATIVE CALCULUS OF PROPOSITIONS

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This paper was originally published in Polish (1939) but in connection with World War II the entire issue was destroyed with the possible exception of two or three copies.[1] Since I have no copy of the Polish edition I have been obliged to restore this paper from memory and there may be inconsequential differences between this and the original. The last paragraph of this version was not included in the Polish paper.

. . . .

The purpose of this paper is to present a complete system of the bi-valued calculus of propositions based on the primitive terms "conjunction" and "negation". This investigation has some interest for the following reasons:

1. It gives a number of theses from the propositional calculus which were formerly unknown or unpublished[2],
2. The axiomatic given for this system differs distinctly from the known axiomatics of the calculus of propositions based on other primitive terms[3],
3. The proof of the completeness of this system is uncommonly simple.

Of equal interest and perhaps more importance, it can be shown that

4. All theses of this system, (i.e., in which no other functors occur except conjunction and negation) are not only provable in the intuitionistic system of Heyting, which was observed by K. Gödel in 1932[4], but also in Minimalcalculus of Johansson, and,
5. Following a recently published result of J. Łukasiewicz[5], the rules of procedure of this system can be proved in the intuitionistic logic, but cannot be proved in Minimalcalculus.

These two last points will be not debated in this paper. In some next issue of this Journal I intend to publish a special paper on these questions.

In the following paragraphs I shall successively derive from the adopted axiomatic such theses as I require, give a proof of the axiomatization and show that each adopted axiom is independent from the others. In the last the axiomatic of the (C;N) – system most resembling that of the axiomatic given will be discussed.

In accordance with the Łukasiewicz's symbolism which is used in this paper the small Latin letters will denote the propositional variables, the capital Latin letters "K", "N", "C" will denote "conjunction", "negation" and "implication" respectively. All functors will be placed ahead of the associated arguments and no parentheses will be used. E.g.: "p and q" is written "Kpq".[6]

All properties of the functors K and N are defined by the following matrix:

(Received February 17, 1953)

I)

K	0	1	N
0	0	0	1
1	0	1	0

in which "1" represents any true proposition and "0" any false. In accordance with the adopted symbolism and this matrix only such expression is senseful which consists of a single variable, or in which the first sign is "N" followed by one senseful expression or in which the first sign is "K" followed by two senseful expressions. Any senseful expression which satisfies the matrix will be called a thesis of the system under investigation.

I shall prove below that any such thesis can be obtained from the following four independent axioms:

1 NKNKNprKNKNqrKNKpqr
2 NKNpKpq
3 NKNqKpKrq
4 NKNKpqNNKpNNq

if we adopt the rules of procedure adequate for the primitive terms (K and N) of this system. There are two such rules:

I. The rule of substitution ordinarily used in the propositional calculus and adjusted to the primitive terms.

II. The rule of detachment adjusted to the adopted primitive terms: "If the expressions "NKαNβ" and "α" are the theses of this system, then also an expression "β" is a thesis of this system."[7]

In order to present this paper compactly I introduce the following symbols and abbreviations:

(a) The Greek letters: α, β, γ and δ will denote any senseful expression.

(b) The Greek letters: ζ_1, ζ_2, ζ_3, . . . will denote either the propositional variables or their negation, i.e., "ζ_1," can denote "p" or "q" or "Np" or "Nr", etc.

(c) The symbol "$\alpha_{(p)}$" will denote that in "α" there occurs one or more times a variables equiform with "p"; consequently "$\alpha_{(-p)}$" will denote that in "α" there does not occur such variable; similarly: "$\beta_{(p)}$", "$\zeta_{1(p)}$", "$\zeta_{2(-p)}$" etc. Also, "$K_{(p)}\alpha\beta$" will denote that in "K$\alpha\beta$" such variable occurs in "α" or in "β" or in both.

(d) It will be said that certain theses belong to the same type, if after replacing their senseful parts by the letters α, β, γ and δ it can be shown that they have similar formal construction. E.g., the axioms 1, 2, 3 belong to the type NKαK$\beta\gamma$, but the axioms 4 does not belong.

(e) The expression "$(\alpha \rightleftarrows \beta) \bullet \{A; B\}$" will denote that in the field of the investigated system using only the theses A and B which belong to this system we can always show that any thesis of the type α is inferentially equivalent with a thesis of the type β. Similarly: "$(\alpha \rightleftarrows \beta; \gamma) \bullet \{A; B; C\}$", etc.

(f) A thesis or an expression, in which no other functors occur except K and N, will be called (K; N)–thesis or (K; N)–expression; similarly: (C)–thesis, (C; K)–expression, etc.

(g) A system being under investigation will be called: (K; N)–system; similarly: (C; N)–system, etc.

1. Using the rules of procedure we can prove from the adopted axioms:

1 NKNKNprKNKNqrKNKpqr
2 NKNpKpq
3 NKNqKpKrq
4 NKNKpqNNKpNNq

the following theses:

4p/NKNpr,q/KNKNqrKNKpqr * NK1 – N5
5 NKNKNprNNKNKNqrKNKpqr
5q/r,r/Kpq * NK2 – N6
6 NKNKNrKpqKNKprKpq
4p/NKNrKpq,q/KNKprKpq * NK6 – N7
7 NKNKNrKpqNNKNKprKpq
7q/Krq,r/q * NK3 – N8
8 NKNKpqKpKrq
5p/q,q/s,r/KpKrq * NK3 – N9
9 NKNKNsKpKrqKNKqsKpKrq
4p/NKNsKpKrq,q/KNKqsKpKrq * NK9 – N10
10 NKNKNsKpKrqNNKNKqsKpKrq
10s/p * NK2q/Krq – N11
11 NKNKqpKpKrq
5p/Kpq,q/s,r/KpKrq* NK8 – N12
12 NKNKNsKpKrqKNKKpqsKpKrq
4p/NKNsKpKrq,q/KNKKpqsKpKrq * NK12 – N13
13 NKNKNsKpKrqNNKNKKpqsKpKrq
13s/Kpq * NK8 – N14
14 NKNKKpqKpqKpKrq
7q/Krq,r/KKpqKpq * NK14 – N15
15 NKNKpKKpqKpqKpKrq
4p/NKpKKpqKpq,q/KpKrq * NK15 – N16
16 NKNKpKKpqKpqNNKpKrq
16p/Np,q/p * NK3p/KNpp,q/p,r/Np – N17
17 NKNpKrp
13s/Kqp * NK11 – N18
18 NKNKKpqKqpKpKrq
10s/NKKpqKqp * NK18 – N19
19 NKNKqKKpqKqpKpKrq
4p/NKqKKpqKqp,q/KpKrq * NK19 – N20
20 NKNKqKKpqKqpNNKpKrq

20q/Np * NK3p/KpNp,q/p,r/Np – N21
21 NKpKrNp
4q/KrNp * NK21 – N22
22 NKpNNKrNp
4p/NKpq,q/KpKrq * NK8 – N23
23 NKNKpqNNKpKrq
23q/NNKrNp,r/q * NK22 – N24
24 NKpKqNNKrNp
4q/Kpq * NK2 – N25
25 NKNpNNKpq
23p/Np,q/NNKpq * NK25 – N26
26 NKNpKrNNKpq
5q/s,r/KrNNKpq * NK26 – N27
27 NKNKNsKrNNKpqKNKpsKrNNKpq
4p/NKNsKrNNKpq,q/KNKpsKrNNKpq * NK27 – N28
28 NKNKNsKrNNKpqNNKNKpsKrNNKpq
28q/NNq,s/q * NK24p/Nq,q/r,r/p – N29
29 NKNKpqKrNNKpNNq
7q/NNKqNNr,r/Kqr * NK29p/q,q/r,r/p – N30
30 NKNKpKqrKpNNKqNNr
4p/NKpKqr,q/KpNNKqNNr * NK30 – N31
31 NKNKpKqrNNKpNNKqNNr
31p/NKNpr,q/NKNqr,r/KNKpqr * NK1 – N32
*32 NKNKNprNNKNKNqrNNKNKpqr
23p/NKpKqr,q/NNKpNNKqNNr,r/s * NK31 – N33
33 NKNKpKqrKsNNKpNNKqNNr
4p/NKpKqr,q/KsNNKpNNKqNNr * NK33 – N34
34 NKNKpKqrNNKsNNKpNNKqNNr
32q/NKqKrs,r/NNKpNNKqNNKrNNs * NK25q/NNKqNNKrNNs – NNK34p/q,q/r,r/s,s/p – N35
35 NKNKpKqKrsNNKpNNKqNNKrNNs
35p/NKNpr,q/NKNqr,r/NKpq,s/r * NK1 – N36
36 NKNKNprNNKNKNqrNNKNKpqNNr
23p/Np,q/Kpq * NK2 – N37
37 NKNpKrKpq
32r/KpKqr * NK2q/Kqr – NNK37p/q,q/r,r/p – N38
38 NKNKpqKpKqr
36p/Kpq,q/r,r/KpKqr * NK38 – NNK3q/r,r/q – N39
*39 NKNKKpqrNNKpKqr
32r/Kpq * NK2 – NNK17p/q,r/p – N40
40 NKNKpqKpq
36q/Kpq,r/Kpq * NK2 – NNK40 – N41
*41 NKNKpKpqNNKpq
41q/Np * NK21r/p – N42
*42 NKpNp
41p/Np,q/p * NK17r/Np – N43

*43 NKNpp
4q/Np * NK42 – N44
*44 NKpNNNp
36r/Kqp * NK17r/q – NNK2p/q,q/p – N45
45 NKNKpqNNKqp
45q/Kpq * NK2 – N46
*46 NKKpqNp
45p/q,q/Kpq * NK17r/q – N47
*47 NKKqpNp
4p/Kpq,q/Np * NK46 – N48
48 NKKpqNNNp
45p/Kpq,q/NNNp * NK48 – N49
49 NKNNNpKpq
36p/NNp,q/Kpq * NK49 – NNK17p/q,r/p – N50
*50 NKNKNNpqNNKpq
50q/KNpq * NK2p/Np – N51
51 NKpKNpq
36r/KNNpq * NK51p/Np – NNK17p/q,r/NNp – N52
*52 NKNKpqNNKNNpq
52p/Nq,q/Kpq * NK17p/q,r/p – N53
53 NKNNNqKpq
36q/NNq,r/Kpq * NK2 – NNK53 – N54
*54 NKNKpNNqNNKpq
4p/Np,q/Kqp * NK17r/q – N55
*55 NKNpNNKqp
39r/NKpq * NK42p/Kpq – N56
56 NKpKqNKpq
4q/KqNKpq * NK56 – N57
*57 NKpNNKqNKpq
32p/q,q/r,r/KpKrq * NK3 – NNK37p/r,r/p – N58
58 NKNKqrKpKrq
36q/Kqr,r/KpKrq * NK2q/Krq – NNK58 – N59
*59 NKNKpKqrNNKpKrq
36q/Kqr,r/KqKpr * NK37q/r,r/q – NNK8p/q,q/r,r/p – N60
*60 NKNKpKqrNNKqKpr
59p/Np,q/r,r/Kpq * NK37 – N61
61 NKNpKKpqr
59p/Nq,q/r,r/Kpq * NK3p/r,r/p – N62
62 NKNqKKpqr
32p/q,q/r,r/KKpqr * NK62 – NNK17p/r,r/Kpq – N63
63 NKNKqrKKpqr
36q/Kqr,r/KKpqr * NK61 – NNK63 – N64
*64 NKNKpKqrNNKKpqr
60p/Np,q/Kpq * NK61 – N65
65 NKKpqKNpr
52p/Kpq,q/KNpr * NK65 – N66

66 NKNNKpqKNpr
36p/NKpq,q/r,r/KNpr * NK66 – NNK17p/r,r/Np – N67
*67 NKNKNKpqrNNKNpr
60p/Nq,q/Kpq * NK62 – N68
68 NKKpqKNqr
52p/Kpq,q/KNqr * NK68 – N69
69 NKNNKpqKNqr
36p/NKpq,q/r,r/KNqr * NK69 – NNK17p/r,r/Nq – N70
*70 NKNKNKpqrNNKNqr
59p/NNKpq,q/Np * NK66 – N71
71 NKNNKpqKrNp
36q/NKqr,r/KpNq * NK2q/Nq – NNK71p/q,q/r,r/p – N72
*72 NKNKpNKqrNNKpNq
59p/NNKpq,q/Nq * NK69 – N73
73 NKNNKpqKrNq
36q/NKqr,r/KpNr * NK2q/Nr – NNK73p/q,q/r,r/p – N74
*74 NKNKpNKqrNNKpNr
32r/KKqpr * NK62p/q,q/p – NNK61p/q,q/p – N75
75 NKNKpqKKqpr
36p/Kpq,q/r,r/KKqpr * NK75 – NNK17p/r,r/Kqp – N76
*76 NKNKKpqrNNKKqpr
59p/NKNpr,q/NKNqr,r/KNKpqr * NK1 – N77
77 NKNKNprKKNKpqrNKNqr
60p/NKNpr,q/KNKpqr,r/NKNqr * NK77 – N78
78 NKKNKpqrKNKNprNKNqr
76p/NKpq,q/r,r/KNKNprNKNqr * NK78 – N79
79 NKKrNKpqKNKNprNKNqr
52p/KrNKpq,q/KNKNprNKNqr * NK79 – N80
80 NKNNKrNKpqKNKNprNKNqr
60p/NNKrNKpq,q/NKNpr,r/NKNqr * NK80 – N81
81 NKNKNprKNNKrNKprNKNqr
59p/NKNpr,q/NNKrNKpr,r/NKNqr * NK81 – N82
82 NKNKNprKNKNqrNNKrNKpq
60p/NKNpr,q/NKNqr,r/NNKrNKpq * NK82 – N83
83 NKNKNqrKNKNprNNKrNKpq
67p/Nq,q/r,r/KNKNprNNKrNKpq * NK83 – N84
84 NKNNqKNKNprNNKrNKpq
70p/Nq,q/r,r/KNKNprNNKrNKpq * NK83 – N85
85 NKNrKNKNprNNKrNKpq
32p/r,q/Nq,r/KNKNprNNKrNKpq * NK85 – NNK84 – N86
86 NKNKrNqKNKNprNNKrNKpq
60p/NKpNr,q/NKNqp,r/NNKpNKqr * NK86p/q,q/r,r/p – N87
87 NKNKNqpKNKpNrNNKpNKqr
67p/Nq,q/p,r/KNKpNrNNKpNKqr * NK87 – N88
88 NKNNqKNKpNrNNKpNKqr
70p/Nq,q/p,r/KNKpNrNNKpNKqr * NK87 – N89

89 NKNpKNKpNrNNKpNKqr
36q/Nq,r/KNKpNrNNKpNKqr * NK89 – NNK88 – N90
*90 NKNKpNqNNKNKpNrNNKpNKqr
23p/Ns,q/KrKqs,r/p * NK3p/r,q/s,r/q – N91
91 NKNsKpKrKqs
32p/r,q/s,r/KpKrKqs * NK37p/r,q/Kqs,r/p – NNK91 – N92
92 NKNKrsKpKrKqs
23p/Nq,q/KrKqs,r/p * NK37p/q,q/s – N93
93 NKNqKpKrKqs
32r/KpKrKqs * NK2q/KrKqs – NNK93 – N94
94 NKNKpqKpKrKqs
36p/Kpq,q/Krs,r/KpKrKqs * NK94 – NNK92 – N95
*95 NKNKKpqKrsNNKpKrKqs
32p/q,q/s,s/KKpqKrs* NK62r/Krs – NNK3p/Kpq,q/s – N96
96 NKNKqsKKpqKrs
32p/r,q/Kqs,r/KKpqKrs* NK37p/r,q/s,r/Kpq – NNK96 – N97
97 NKNKrKqsKKpqKrs
36q/KrKqs,r/KKpqKrs * NK61r/Krs – NNK97 – N98
*98 NKNKpKrKqsNNKKpqKrs

2. Let us suppose that there are some theses which satisfy the given matrix and which are independent of the adopted axiomatic. Among them there must be one thesis which is shortest. I shall show that there is not such a "shortest independent thesis" and, therefore, any thesis which satisfies the given matrix is a consequence of the adopted axiomatic. The proof is as follows:

2.1 No expression which consists of the small Latin letters, e.g., p, q, etc. or small Latin letters preceded by optional numbers of negations, e.g., Np, NNq, etc. satisfies the given matrix and, therefore, such expressions cannot be theses in this system.

2.1.1 Thus, any thesis in this system must belong to one of the three following types:

T I $K\alpha\beta$; T II $NN\alpha$; T III $NK\alpha\beta$

But, any thesis of T I can be obtained from two shorter theses, as ($K\alpha\beta \rightleftarrows \alpha; \beta$) * {46; 47; 57} and any thesis of T II can be obtained from a shorter thesis, as ($NN\alpha \rightleftarrows \alpha$) * {43; 44}. Therefore, any thesis of T I to T II can be proved from the theses of T III.

2.1.2 The theses of T III belong to one of the four following types:

T1.1 $NKNN\alpha\beta$; T1.2 $NKK\alpha\beta\gamma$; T1.3 $NKNK\alpha\beta\gamma$; T1.4 $NK\zeta_1\beta$

Any thesis of T1.1 can be obtained from a shorter thesis of T1.1, T1.3 or T1.4, as ($NKNN\alpha\beta \rightleftarrows NK\alpha\beta$) * {50; 52}, and any thesis of T1.2 can be obtained from a thesis of T1.3 or T1.4, as ($NKK\alpha\beta\gamma \rightleftarrows NK\alpha K\beta\gamma$) * {39; 64} and any thesis of T1.3 can be obtained from two shorter theses, as (NKNK

$\alpha\beta\gamma \rightleftarrows NKN\alpha\gamma; NKN\beta\gamma) * \{67; 70; 32\}$. Thus, it is evident, that any thesis of T1.1, T1.2 or T1.3 can be proved from some not longer theses of T1.4.

2.1.3 The theses of T1.4 belong to one of the five following types:

T2.1 $NK\zeta_1 NN\beta$; T2.2 $NK\zeta_1\zeta_2$; T2.3 $NK\zeta_{1(p)}\beta_{(-p)}$

T2.4 $NK\zeta_1 NK\alpha\beta$; T2.5 $NK\zeta_{1(p)}K_{(p)}\alpha\beta$

Any thesis of T2.1 can be obtained from a shorter thesis of T2.2 – T2.5, as $(NK\alpha NN\beta \rightleftarrows NK\alpha\beta) * \{54; 4\}$ and, if an expression of T2.2 satisfies our matrix I, i.e., it is a thesis of this system, then it is only either the thesis 42 or the thesis 43 and if an expression of T3.3 satisfies the matrix I, then also the shorter expression of the form $N\beta_{(-p)}$ must be a thesis of this system. But, evidently any thesis of the form $N\beta_{(-p)}$ we can get from some not longer theses of T2. And, due to the thesis 55 we can always obtain from the thesis of this form $N\beta_{(-p)}$ a previous thesis of T2.3. Any thesis of T2.4 can be obtained from two shorter theses, as $(NK\alpha NK\beta\gamma \rightleftarrows NK\alpha N\beta; NK\alpha N\gamma) * \{72; 74; 90\}$. Thus, any thesis of T2.1 – T2.4 either is a consequence of the adopted axiomatic or can be proved from some shorter theses and, therefore, if there exists the "shortest independent thesis", then it must belong to T2.5.

2.1.4 The theses of T2.5 belong to one of the two following types:

T2.51 $NK\zeta_{1(p)}K\alpha_{(-p)}\beta_{(p)}$; T2.52 $NK\zeta_{1(p)}K\alpha_{(p)}\beta$

Any thesis of T2.51 we can reduce to the some not longer thesis of T2.52, as $(NK\alpha K\beta\gamma \rightleftarrows NK\alpha K\gamma\beta) * \{59\}$ and any thesis of T2.52 can be obtained from some not longer thesis of the type:

T3 $NK\alpha_{(p)}K\zeta_{1(p)}\beta$

as $(NK\alpha K\beta\gamma \rightleftarrows NK\beta K\alpha\gamma) * \{60\}$. The theses of T3 belong to one of the five following types:

T3.1 $NKNN\alpha_{(p)}K\zeta_{1(p)}\beta$; T3.2 $NKK\alpha\beta_{(p)}K\zeta_{1(p)}\gamma$

T3.3 $NKK\alpha_{(p)}\beta K\zeta_{1(p)}\gamma$; T3.4 $NKNK_{(p)}\alpha\beta K\zeta_{1(p)}\gamma$;

T3.5 $NK\zeta_{2(p)}K\zeta_{1(p)}\gamma$

But, any thesis of T3.1 can be obtained from a shorter thesis of T3.2 – T3.5, as $(NKNN\alpha\beta \rightleftarrows NK\alpha\beta) * \{50; 52\}$ and any thesis of T3.2 can be reduced to some not longer thesis of T3.3, as $(NKK\alpha\beta\gamma \rightleftarrows NKK\beta\alpha\gamma) * \{76\}$. Any thesis of T3.3 we can always reduce to the some not longer thesis of T3, as $(NKK\alpha\beta K\gamma\delta \rightleftarrows NK\alpha K\gamma K\beta\delta) * \{95; 98\}$, but in which its first member "$\alpha_{(p)}$" is shorter, than the first member of the original thesis, i.e., "$K\alpha_{(p)}\beta$". Therefore, it is evident, that using suitable times the above given methods we can always reduce any thesis of T3.1 – T3.3 to the some not longer thesis of T3.4 or T3.5. Thus, any thesis under investigation either is a consequence of the adopted axiomatic or belongs to T3.4 – T3.5.

But, any thesis of T3.4 can be obtained from two shorter theses of:

T4 $NKN\alpha K\zeta_{1(p)}\beta$

as ($NKNK\alpha\beta\gamma \rightleftarrows NKN\alpha\gamma$; $NKN\beta\gamma$) * {67; 70; 32}.

2.1.5 Any thesis of T4 belong to one of the three following types:

T4.1 $NKN\alpha_{(-p)}K\zeta_1\beta_{(p)}$; T4.2 $NKN\alpha_{(-p)}K\zeta_{1(p)}\beta_{(-p)}$

T4.3 $NKN\alpha_{(p)}K\zeta_{(p)}\beta$

Any thesis of T4.1 can be reduced to the same not longer thesis of T3, as ($NK\alpha K\beta\gamma \rightleftarrows NK\gamma K\beta\alpha$) * {59; 60} and, consequently, it either is a consequence of the adopted axiomatic or belongs to T3.5 or can be proved from some shorter thesis of T4. Any thesis of T4.2 can be reduced to the some not longer thesis of T2.3, as ($NK\alpha K\beta\gamma \rightleftarrows NK\beta K\alpha\gamma$) * {60} and, consequently, can be obtained from a shorter thesis. The theses of T4.3 are theses of T3, but shorter than the original theses, and, therefore, as it was shown avove, can be obtained either from our axiomatic or from the theses of T3.5 or from shorter and shorter theses of T3. Hence, any thesis of T4 and then any thesis of T3.4 either is a consequence of the adopted axiomatic or can be proved from the theses of T3.5.

2.1.6 The theses of T3.5 belong to one of the three following types:

T3.51 $NK\zeta_{1(p)}K\zeta_{1(p)}\beta$; T3.52 $NKpKNp\beta$;

T3.53 $NKNpKp\beta$

Any thesis of T3.51 can be obtained from a shorter thesis, as ($NK\alpha K\alpha\beta \rightleftarrows NK\alpha\beta$) * {41; 55} and any thesis of T3.52 can be reduced to some not longer thesis of T3.53, as ($NK\alpha K\beta\gamma \rightleftarrows NK\beta K\alpha\gamma$) * {60}. But any thesis of T3.53 is a substitution of the thesis 2.

Therefore, it is evident, that any thesis of this system either is a consequence of the adopted axiomatic or can be gradually deduced from some shorter and shorter theses. Then, because any thesis possesses only a finite number of the signs, there are no theses which satisfy our matrix I and are not a consequence of the adopted axiomatic. Hence, there is not a "shorter independent thesis", and, therefore, the given axiomatic is adequate for our matrix.

2.2 Now, let us assume that an expression:

(a_1) $\Psi(p, q, r, \ldots)$

symbolizes any senseful (K; N)–expression which is not a thesis of this system, i.e., it is a such expression that for some substitutions of 0 and 1 for its variables p, q, r, . . . this expression α_1 does not satisfy the given matrix I. Having this expression α_1, we have also the following expression:

(α_2) Ψ (p, NKpNp)

which can be obtained from the expression α_1 in the following way: Knowing that some substitutions of 0 for some variables and some substitutions of 1 for another variables show that the expression α_1 is not a thesis of (K; N)–system, we substitute for every variable for which before we substituted 0 a variable "p" and for any other variable – "NKpNp". Then, evidently, the expression α_2 does not satisfy the matrix I, if we shall substitute for p/0. But, an expression:

(α_3) NK Ψ (p, NKpNp) Np

is a thesis of the (K; N)–system, because for any substitution of 0 and 1 for its sole variable "p" it satisfies the matrix I. And, therefore, it can be deduced from the adopted axiomatic. But, from the expressions α_2 and α_3 we get at once an expression:

(α_4) p

from which we can derive any senseful expression. Therefore, our system possesses the 2nd degree of extension.[8]

Thus, any thesis of this system is a consequence of the adopted axiomatic and any extension of this axiomatic gives a contradictory.

3. The following matrices II–V which verify the rule of detachment of (K; N)–system show the adopted axioms are mutually independent.

3.1 The independency of the axiom 1 is proved by a matrix:

II) *

K	0	1	2	3	N
0	0	0	0	0	1
1	0	1	2	0	0
2	0	3	1	0	3
3	0	0	0	0	1

in which the designated value is 1. This matrix satisfies the axioms 2, 3 and 4, but not 1, e.g., if we substitute in 1 for p/2, for q/1 and for r/1: NKNKN21 KNKN11KNK211 = NKNK31KNK01KN31 = NKN0KN0K11 = NK1K11 = NK11 = N1 = 0.

3.2 The independency of the axiom 2 is proved by a matrix:

III)

K	0	1	2	N
0	0	0	1	1
1	0	1	1	0
2	0	1	1	0

in which the designated value is 1. This matrix satisfies the axioms 1, 3 and 4, but not 2, e.g., if we substitute in 2 for p/0 and q/2: NKN0K02 = NK11 = N1 = 0.

3.3 The independency of the axiom 3 is proved by a matrix:

IV)

K	0	1	2	N
0	0	0	0	1
1	0	1	1	0
2	1	1	1	0

in which the designated value is 1. This matrix satisfies the axioms 1, 2 and 4, but not 3, e.g., if we substitute in 3 for p/1, for q/0 and for r/2: NKN0K1K20 = NK1K11 = NK11 = N1 = 0.

3.4 The independency of the axiom 4 is proved by a matrix:

V)

K	0	1	2	3	N
0	0	0	0	0	1
1	0	1	0	2	0
2	0	2	0	0	3
3	0	2	0	0	1

in which the designated value is 1. This matrix satisfies the axioms 1, 2 and 3, but not 4, e.g., if we substitute for p/3 and for q/2: NKNK32NNK3NN2 = NKN0NNK3N3 = NK1NNK31 = NK1NN2 = NK1N3 = NK11 = N1 = 0.

4. The given above axiomatic for this (K; N)–system differs distinctly from the known axiomatic of the calculus of propositions based on the other primitive terms. In the field of the (C; N)–system of the calculus of propositions, perhaps, the most resembling axiomatic to the given above is the following:

Q1 CCNprCCqrCCpqr
Q2 CNpCpq
Q3 CqCpCrq

4.1 Using the rules of procedure adjusted to this system we can easily prove from the axioms Q1 – Q3:

Q1q/r,r/CpCqr * CQ2q/Cqr – CQ3q/r,r/q – Q4
Q4 CCprCpCqr
Q4p/Np,q/r,r/Cpq * CQ2 – Q5
Q5 CNpCrCpq
Q1p/q,q/Cpr,r/CpCqr * CQ5p/q,q/r,r/p – CQ4 – Q6
Q6 CCqCprCpCqr
Q6p/CNpCpq, r/Crq * CQ3p/CNpCpq – CQ2 – Q7
Q7 CqCrq
Q6p/CqCrq,r/q * CQ7r/CqCrq – CQ7 – Q8
Q8 Cqq
Q6q/Np,r/q * CQ2 – Q9
Q9 CpCNpq
Q6q/CNpq,q/Cqq,r/CCpqq * CQ1r/q – CQ8 – Q10
Q10 CCNpqCCpqq
Q6p/Cpp,q/CNpp,r/p * CQ10q/p – CQ8q/p – Q11
Q11 CCNppp
Q6p/q,q/Cqr,r/Cpr * CQ4p/q,q/p – Q12
Q12 CqCCqrCpr
Q1r/CCqrCpr * CQ5q/r,r/Cqr – CQ12 – Q13
Q13 CCpqCCqrCpr

Thus, from the axioms Q1 – Q3 we have obtained, having the theses Q13, Q11 and Q9, the well-known axiomatic of the propositional calculus of Łukasiewicz.[9]

4.2 The independency of the axioms Q1 and Q2 is obvious. The following matrix VI in which the designated value is 1 shows that Q3 is independent from Q1 and Q2:

VI)

C	0	1	2	3	N
0	1	1	1	1	1
1	0	1	2	0	0
2	1	1	1	1	3
3	0	1	0	0	2

This matrix satisfies the axioms Q1 and Q2, but not Q3, e.g., if we substitute for p/3, for q/3 and for r/3: C3C3C33 = C3C30 = C30 = 0.

4.3 The given above axiomatic: Q1, Q2, Q3 of the bi-valued (C; N)-calculus can be considered, as a simplification of the following Łukasiewicz's axiomatic:[10] Q1 and

Q2* CCCpqrCNpr
Q3* CCCpqrCqr

NOTES

[1] Cf. Sobociński$_1$ and Sobociński$_2$, pp. 10 – 11. The paper Sobociński$_1$ can be known only by a detailed review of Prof. H. Scholz, cf. Scholz$_1$. See also *The Journal of Symbolic Logic*, vol. VI, p. 72 and p. 115. Cf. also Prior$_1$, p. 5, and Łukasiewicz$_3$, p. 202.

[2] E.g. in Whitehead – Russell$_1$ there is only one such thesis *3.24, i.e. our thesis 42.

[3] Cf. par. 4 of this paper.

[4] Cf. Gödel$_1$, see also Łukasiewicz$_3$, p. 202.

[5] Cf. Łukasiewicz$_3$.

[6] The explanation of this symbolism can be found, e.g., in Łukasiewicz$_1$, pp. 77 – 83. The given below forms of the proofs of the theses are used usually in the papers written in this symbolism, cf. Sobociński$_3$, pp. 24 – 32. Here these forms are adjusted to the primitive terms of this system. The theses which appear with an asterisk will be used in the next considerations.

[7] This rule corresponds exactly to the rule of detachment generally used for implication. This correspondence is a simple consequence of the universally known equivalence of the expressions "C$\alpha\beta$" and "NKαNβ". It is evident that a similar rule:

> "If the expressions "NK$\alpha\beta$" and "α" are the theses of this system, then also an expression "Nβ" is a thesis of this system."

is not sufficient. Having such a rule we are unable to prove the theses of the form "K$\alpha\beta$", e.g. the thesis 51 below.

We omit the rule of definitions, because this rule is unimportant for our present investigations.

[8] The term "degree of extension" is used in the sense of the Definition 6 from Tarski$_1$.

[9] Cf. Łukasiewicz$_1$, p. 80, and Łukasiewicz$_2$.

[10] Cf. Łukasiewicz$_2$, the remarks to the theses 59, 19 and 37.

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THE RELATIONS BETWEEN LOGICAL, MATHEMATICAL AND COMPUTING MACHINE SYSTEMS

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INTRODUCTION

Computing machines are usually thought of as being devoted essentially to the solution of mathematical problems and the terms used to describe their principal operations are derived from arithmetic. Yet most of the basic circuitry is commonly discussed in a vernacular that has its source in logical systems. It is an interesting analogy with the fact that logic provides the foundations for mathematics. The literature of computing machinery contains a great deal of material pertaining to logical systems and references to Boolean Algebra, Propositional Calculus and the like are very common. Comparisons between logical, arithmetic, mechanized and living systems appear at all levels of thought in this field.

Until the advent of high speed computers a little more than a decade ago, the study of logical systems did not have obvious practical significance. Consequently many people active in the design of computers have lost track of the academic pattern of logical thought. This short article is intended to provide a broad discussion of the relationships mentioned in the title and to trace a few important threads in the cloth of history. It is written largely in response to requests and suggestions received during the last year.

Everything that follows has been presented in more complete and satisfactory form elsewhere by others. If this material has utility it is because it may provide a quick review that illuminates certain commonly used references and underscores special characteristics of some systems that are now and then misunderstood. In order to provide emphasis and also to develop some of the information from more than one point of view there is considerable redundancy.

I.

A system consists of a set of restrictions applied to a set of elements. The restrictions in a system include the way in which the elements are related. A system may have restrictions as its elements. We may consider a system of logic as a precise language for communicating in an exact way. Any system of communication must include restrictions about definitions. The restrictions in a logical system are a set of rules and the elements are words.

A deductive system of logic consists of a set of axioms and a set of rules which in combination are capable of generating expressions that have sense in the system. A semantic system applies restrictions to the meaning of words. Any symbols may be used to represent words, or numbers or objects of any kind and the character (logical, philosophical, mathematical etc.) of a system is not uniquely determined by its symbolic forms.

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A mathematical system concerns restrictions about elements representing quantities. In a logical system the elements may represent quantities, qualities, objects, actions, concepts or anything that may be a subject of communication.

An arithmetic system concerns restrictions about numbers. With the exception of certain special systems, arithmetic may be entirely defined by addition. Addition may be defined in strong systems of logic. All systems of the kind described have a logical foundation.

A static system has no reference in time. A computing machine is a device capable of dynamic implementation of some static system. It is known that certain primitive operations of the brain are similar to the performances of computing machines. The brain is a physical structure observable by sensing devices that communicate with it. The mind of any individual includes the brain and may also include additional structures that may or may not be observable or even conceivable by the brain or the mind. It is probably true that the mind will someday fully understand the brain. It is not known whether the capability fully to understand itself is either inherent or latent in the mind. All known systems and machines are known because they were discovered and developed by some mind(s).

The mind and brain manipulate elements normal to their heredity and function, as well as elements consisting of observations communicated by various sensing systems. The ecology of the organism is observed through various sensing systems and everything outside is called the world. The mind tends to manipulate all observations as though they were different aspects of one world. The mind strives to understand and define the world. Logical systems are studied, pursued and developed by the mind in an effort to understand, define and describe the world.

II.

The Calculus of Propositions is a system for analysis of the truth value of declarative propositions and the linkages between such propositions and it has nothing to do with the internal meaning or internal linkages within such propositions. The Calculus of Name (or Terms) and the Calculus of Relations have to do with the relations between the names of objects to which names are appropriate including general, individual and empty names. By means of the latter two systems it is possible to define the notions involved in arithmetic and mathematics and to arrive at an understanding of the basic framework that forms the foundations of mathematics.

Boolean Algebra is a system which defines some elementary connections among elements in a given universe. No assumptions are made about these elements except that they belong to the given universe and that this universe contains at least two such elements. Thus many interpretations of Boolean Algebra (and similar theories) are possible and the level of abstraction is such as to permit application to various branches of science, and particularly to the field of mathematics. Boolean Algebra may be interpreted as a Calculus of Propositions and has been useful in expressing a great many basic notions associated with the design of computing machinery.

So-called Switching Algebra, Decision Element Logic and the like constitute simple interpretations of the Calculus of Propositions and Boolean Algebra in terms and symbols that are adapted to computing machine structure – their contribution is principally one of convenience and perhaps improved special purpose symbolism.

Modal Logic describes systems that include such expressions as "It is possible that....." and "It is necessary that....." and the most well known such systems are extensions of the Calculus of Propositions. The concepts of Modal Logic were by no means unknown to Aristotle and they are mentioned in various of his writings. There have been, and are, many opinions that modalities and time are linked. The expression, "It is possible that....." may be interpreted as "It is possible that at some future time....." and has been so considered by various early logicians. These questions are discussed at some length by A. N. Prior in a paper to be published in the next issue of this *Journal.*

Time is a factor of no small significance in connection with the operation of computing machinery. It has been pointed out that there are several functors that cannot be individually used to define a complete system in Propositional Calculus but each may be used as the sole primitive term of Decision Element Logic because of the introduction of time as a parameter of the system. Many computing machine circuits are impossible to define without reference to time because they include feedback connections in which the output is returned in whole or in part to the input and may aid or degrade the input signal.

Adequate analogies in existing logical systems suitable for describing sequentially timed operations in computing circuits are not readily constructed. An expression in the Calculus of Propositions exists without reference to time while the same expression applied to computing machine logic implies a succession of events, each of which is dependent upon the completion of some earlier event. Thus the expression "If p and.q then r and s." as applied to a computing circuit describes a gate through which a pulse may pass upon the conjunction of r and s provided that there is also a conjunction between p and q, or still more difficult of interpretation, provided that there *has been* a conjunction of p and q.

Systems of control in computing machines, in living organisms and probably in all dynamic systems include some form of feedback. Any language that is to be adequate fully to describe the world must include expressions suitable for dealing with these problems.

Almost all contemporary computing machines are aggregates of comparatively simple devices and their greatest utility is in the rapidity with which they can perform addition – hence all ordinary arithmetic operations. Computing machines may be properly thought of as the mechanical embodiment of logical, arithmetic or mathematical systems depending upon the restrictions that define their operation, their symbols and the way in which their operations are interpreted. To the extent that it is true that all problems of analysis may be solved by arithmetic means, it is possible to perform the brute force operations of analytical problems at high speed and with great convenience using computing machines. A contemporary computing machine may be thought of as a desk type calculating machine with a built-in operator, the latter being extremely fast but quite stupid.

There are many appropriate analogies between computing machines and the human brain, for indeed the living organism contains structures capable of performing the same routine operations handled by computers, (much more efficiently but somewhat slower). The important differences are more obvious and basic. So far no computing machine has made an invention, designed a new system or solved a wide variety of problems in the broad field for which it was not specifically programmed. Many computing machines have been designed to play games with great skill but no computer has yet suggested a new game.

Investigations in the field of electrophysiology during recent years have produced a great deal of basic information and more is undoubtedly forthcoming. It is interesting and important for cross-interpretations of logical systems, mathematical structures, computing machine designs and living organisms to be made, and such comparisons can be a rich source of increased understanding.

III.

The following portion of this article consists of a brief historical outline intended to provide an evolutionary framework for the understanding of various contemporary points of view. The material is developed with a particular perspective in mind; it is far from detailed or complete and many illustrious names and important discoveries are not even mentioned.

In the fourth century before Christ the Greek philosopher Aristotle constructed the earliest known deductive system of logic in a very precise and beautiful way. The basic system of Aristotle established certain relations between general, unempty names. Individual names and empty names were excluded from his system. Most misunderstanding about the validity of Aristotle's form of reasoning arise from not recognizing these restrictions. Thus any syllogism about, for example Socrates, is an improper example because "Socrates" is an individual name. In various special cases such syllogisms may happen to be valid but they are still excluded from the deductive reasoning system originated by Aristotle.

There were two forms of reasoning in Aristotle's system. One of these expressed the relations between two propositions which, for example, might involve reasoning from the general to the particular, as "If all horses have tails then some horses have tails." The other form expresses the relations between three propositions in which from two premises it is possible to deduce a conclusion.

Aristotle always used such forms of expressions as:

"If p then q." or "If p and q, then r."

Aristotle did not develop a Calculus of Propositions but he knew several of the laws of such systems.

Approximately a hundred years after Aristotle the Stoic school of philosophy introduced a new kind of logic which consisted of a Calculus of Porpositions expressed in a special pattern. Thus the Stoics used the form:

p
if p, then q
∴ q

In the latter era of antiquity the systems of Aristotle and the Stoics were mixed and confused. The Calculus of Propositions was dropped from logic but the Stoic patterns and forms were retained and introduced into the Syllogistic of Aristotle. Thus, since that time, the syllogisms are conventionally presented in the form:

A
B
∴ C

This mixture, which retained obscure elements of the Calculus of Propositions, was developed around the fifth or sixth century after Christ into a system later called Classical (or Traditional) Logic.

During the Middle Ages several logicians, philosophers and mathematicians discovered new laws that are now important in Modern Formal Logic. In the fourteenth century Traditional Logic had in fact been expanded to the point where a new approach was on the horizon but for various evolutionary reasons this movement never came to fruition. Scholastic Philosophy was dropped during the Renaissance and the most significant logical writings of the Scholastic Philosophers, such as William of Occam, were never circulated. Printing was invented during this period but nowhere in the incunabula are important contributions of the then contemporary logicians found. Considerable material of a philosophical and theological nature, such as the writings of St. Thomas Aquinas, was published but although these books contain complicated reasoning systems they were not essentially devoted to logical structures.

In the beginning of the seventeenth century Leibniz knew that Traditional Logic was inadequate and he discovered several logical laws but his writings in this field were not published until the twentieth century (by Louis Couturat). Also in the seventeenth century Arnould, with the help of the authors and members of Port Royal published the famous *La logique du Port Royal* which became a model for future text books in Traditional Logic and practically eliminated any further developments in the field of logic for almost two hundred years.

Not until the nineteenth century did mathematicians widely recognize the serious limitations of Traditional Logic for purposes of their research work. Then many independent sources produced new logical systems. One of the earliest of these new logics was presented by George Boole. He published a system concerned with relations between "elements" which can be interpreted as some form (theory) of addition. Boolean Algebra deals with relations between undefined elements which may be interpreted as general (unempty and empty) names. In the Boolean system, because of the broad inclusion of empty names, certain of Aristotle's syllogisms are not valid. In the same general period De Morgan, Sir William Hamilton and others introduced new logical concepts and the outline of Modern Formal Logic began to develop.

Peirce introduced a Calculus of Relations in relatively simple form and this was elaborated on by Ernst Shrōder. Independently, in his early writings, Gottlob Frege gave the first adequate and precise outline of the Calculus of Propositions. Also independently, but during the same approximate time, Georg Cantor introduced Set Theory where a "set" is treated as a "set of some objects." Cantor developed the relations between these "sets" and their elements. In the face of great initial opposition he eventually convinced mathematicians that these notions have importance. Cantor defined so-called Cardinal numbers and the relations between them and first introduced the notion of "ordered" and "well-ordered" sets. He also gave a profound analysis of the properties of infinite sets. Although both Galileo and Bolzano had also contributed to the ideas it remained for Dedekind to provide a precise analysis of infinite sets toward the end of the nineteenth century.

In the nineteenth century there grew an increasing tendency to explore the arithmetization of all notions that occur in analysis, and it was eventually demonstrated that all such questions, (including, for example, all problems in the fields of differential and integral calculus) may be expressed in arithmetic terms and resolved by arithmetic operations. Several mathematicians have shown that it is possible to define all numbers using only positive integer numbers. Sir William Hamilton demonstrated that complex numbers may be expressed as pairs of real numbers, and so on. K. Weierstrasse showed that all notions of mathematical calculus can be defined by arithmetic methods. Kronecker suggested that "God created positive integer numbers and all other numbers are designed by man."

Considerable attention was drawn to the problem as to whether the integer numbers are self-independent mathematical notions or whether they can be reduced to some logical concepts. It is now known that the latter can be accomplished if we define each positive integer as the name of the class of all finite classes comprising exactly that number of elements. This is a definition for a finite Cardinal number and is a particularly high level of abstraction since it does not relate to the nature of the elements nor to their order, but only to their total number.

Gottleb Frege attempted to develop a logical system which all of mathematics could be deduced – a complete logical foundation for mathematics. He tried in two ways. The first from an essentially philosophical point of view and the second a systematical approach. At the time that the reference (18) had been published and the reference (19) was at the printers Frege received a letter from the then young British logician – Bertrand Russell. This letter pointed out a contradiction in the system of Frege. The antinomy of Russell can be constructed not only in Frege's work but also in other systems. Before Russell a different antinomy had been found in Cantor's set theory by Burali-Forti and still another was discovered by Cantor himself. Cantor's reasoning method was intuitive and he never gave an axiomatic for set theory but the Russell antinomy may be constructed in his system. Russell was the first strongly to underline the question of the antinomies.

The great problem before all mathematicians and logicians at the turn of the twentieth century was whether a foundation for mathematics could be built with no antinomy. The difference between a simple contradiction and an antinomy must be clearly distinguished. In accordance with L. Nelson an antinomy is described as a contradiction which arises from assumptions which are believed to be true and from reasoning methods in which there is confidence and faith. Thus any antinomy involves a psychological conflict. A simple contradiction can be dropped by some modification of the system in which it occurs. An antinomy can only be resolved properly by destroying belief in the assumptions and reasoning processes that lead to it. Most efforts to resolve the antinomies have been in applying special restrictions to limit the system involved to that the antinomies cannot be constructed. This method has artificial qualities that are not entirely satisfactory from a subjective, intuitive point of view.

At the beginning of this century there appeared two systems, one mathematical and one logical. Ernst Zermelo rebuilt the set theory of Cantor giving an axiomatical approach in such a way that neither the Russell or other antinomies can be constructed. In effect Zermelo forbids the definition of certain kinds of sets in order to make it impossible to obtain the known antinomies. These restrictions narrow the means of definition significantly. Zermelo succeeds in avoiding but not in resolving the antinomies. The axiomatic set theory of Zermelo was modified by Adolf Fraenkel and separately by J. von Neumann and in all of these forms is an essentially mathematical structure even though it is constructed in a way that is similar to a logical system.

The other method of resolving the design of a foundation of mathenatics without antinomies was developed by Russell and introduced as the Theory of Logical Types. Russell said that logical notions belong to various simple and more complicated types and an antinomy results when the types are improperly mixed. An inadequate and imprecise analogy may be made with the parts of speech for purpose of brief explanation. A verb cannot be substituted for a noun without introducing confusion and the idea of logical types is loosely similar. The Russell Theory of Types in the final form of presentation in *Principia Mathematica* was called the Ramified Theory. From the logical point of view Type Theory is a more natural resolution of the problem of antinomy than Set Theory but for many mathematical purposes Type Theory is quite inconvenient.

There have been a number of modifications of Type Theory, perhaps the most notable of which has been given by Chwistek and widely adopted. An entirely different modification has been developed by Quine somewhat in the direction of Set Theory concepts. There have been several variations of the Quine Stratification systems.

An important, but not widely understood, "foundation for mathematics" was developed by Lesniewski with the purpose of eliminating the antinomies in a way that explains why they cannot be constructed rather than avoiding them with arbitrary limitations. This theory is rather nearer to the Simple than to the

Ramified Type Theory but there are essential differences. He introduced logical types which he called Semantical Categories, primarily for purposes of introducing order to the system – which also resolved the problems of the known antinomies.

IV.

It is a characteristic of modern systems that lean toward logical constructions to use some form of type theory, e.g. Leśniewski's Ontology, and those of a more mathematical nature toward set theory, e.g. Quine's Stratification. The principal interest of researchers in these fields today was inauguarated by D. Hilbert in the developement of metalogics, i.e. theories which study systems of logic and mathematics as their objects. The problems belonging to metalogic and metamathematics are very complicated and difficult and only a few of the simpler questions are mentioned in the following paragraphs.

It is of consequence to find methods of proof as to whether a given system is non-contradictory. For several logical systems we have such proofs but, for example, we do not know either way about Set Theory and the systems of Quine.

Problems of completeness are significant. In accordance with the following definition:

A system is complete if any senseful expression is either the consequence of the axiomatic or if added as a new axiom gives a contradiction.

the bi-valued Calculus of Propositions is complete but no broad logical system nor any system of arithmetic is complete. Furthermore, K. Gõdel and others have given proofs that no strong system of logic or mathematics can be so constructed. No matter how strong the axiomatic of such systems it is always possible to find a thesis which is independent and when added as a new axiom does not give a contradiction. There are, of course, other definitions of completeness which have importance in more profound analyses of metalogical questions.

The so-called "problem of decision" has great interest. This involves the question whether, having some axiomatic system, it is possible to develop a strict method by means of which one can prove any thesis which has sense in the system. It has been demonstrated that only a few very simple systems allow such resolution. In computing machine terms the question is whether any machine, no matter how complicated, can resolve any and all problems in the field of its system. For most systems such a machine appears to be impossible. It may be worthy of emphasis that the point here does not merely involve establishing a program capable of solving a specific given problem but rather designing a machine capable of *self-programming* for any and every problem appropriate to a given system. Many people are investigating such designs and it is of interest to understand the limitations as well as the advantages of potential developments.

From the beginning of logical thought there has been a great deal of intuitive support for the thesis that any declarative proposition must be either true or false. Post and Łukasiewicz conducted independent investigations to analyze the effect of introducing many valued variables into logical structures. The results give new systems, quite different from bi-valued propositional logic. Some investigators have shown that it is possible to build many-valued functional calculus. A wide variety of systems have been studied and infinite valued logic is possible. Non-contradictory many-valued systems have been demonstrated. Whether these many-valued logics are useful for research and whether they may be adopted for some description of the real world has not yet been determined. At present such structures exist almost entirely in terms of abstract discipline.

A variety of factors have contributed to the rapid development of digital computing machinery during the last decade, of which the most important are influence by the activities of many government agencies. Most of these machines perform very simple basic operations at extremely high speeds using very large numbers. Their advent has given even greater impetus to the implementation of analysis via arithmetic.

Partially because it is mechanically and electrically convenient to design devices having only two stable states these machines usually operate in binary arithmetic. The processes involved in binary arithmetic are entirely similar to the operations indicated in the bi-valued Calculus of Propositions. In the early development of machine methods the interpretations possible between bi-stable machine elements, binary arithmetic and bi-valued logic became increasingly obvious. Actually the elements of machines have characteristics that may be combined so as to perform operations that are either essentially logical or mathematical or at least may be most readily interpreted as one or the other. Most practical machines today are designed from the analytical arithmetic point of view but a wide variety of special devices aimed at handling logical expressions exclusively have been experimentally produced.

Computing machines are generally thought of as operating in the field of mathematics. Clearly, any strength or weakness in the logical foundations of mathematics is reflected in existing and potential characteristics of computing machines and any knowledge in these fields may be interchangeably useful. It is desirable that there be increased interchange of thought between logicians, mathematicians and computing machine designers.

Basic elements in computers such as "coincidence" and "mixer" circuits were early recognized as similar to logical "and", "or" and so on. Boolean Algebra has been a useful tool for analysis and the Decision Element systems and notations are direct interpretations from the Calculus of Propositions. But these are all more or less obvious ideas and a great deal of work is yet to be done in exploring more profound and important relationships.

Among contemporary developments of significance is the so-called Information Theory which originated in the work of Norbert Wiener and the subsequent classical investigations of C. E. Shannon. Information Theory is concerned with the quantitive evaluation of signals that serve to transmit communicable ideas. The fact that in a particular language the number of possible different messages having a given symbol length varies directly and exponentially with the message length is important in the current form of the theory. A major portion of the significant work in the field of Information Theory is associated with the study of restrictions applied to a set of symbols that cause such a structure to be capable of functioning as a language. In the context of Information Theory a quantum of information is transmitted under conditions where the receiving structure is notified by the transmitting structures with regard to which value is correct for a given binary argument. This unit of information is termed a "bit" or preferably "binit" from "binary digit." The problems of Information Theory are somewhat complicated by the quantitive dependence of the measure of information on the receptor. This is not analogous but is somewhat akin to the condition in living organism's where the interpretation on input data is a function of the physical connection of a nerve fiber to a specific locale in the central structure. For example if fibers from the 8th cranial nerve were severed from the auditory paths and connected to the visual cortical centers some sounds would generate visual responses. These facts lead to a wide variety of fascinating neuro-surgical speculations.

V.

It is the function of a philosopher to describe the outside world to himself, and to others, and to explain the meaning of this world. Aristotle was a philosopher. He became the first logician in his efforts to develop a system suitable for explaining his philosophy. For nearly two thousand years his basic structures were modified, changed and often confused but not significantly expanded. In the one period when such an expansion might have grown the voices of the prophets of Scholastic Philosophy were lost in the Renaissance and not until the middle of the nineteenth century were important new contributions made.

Modern Formal Logic (Symbolic Logic) is primarily concerned with the design and construction of precise languages for the description of the world and in the foundation of reasoning processes involved in mathematics. Mathematics is a structure that deals with the notions that can be described quantitatively and with the methods by means of which it is possible so to manipulate such measures that useful information may be obtained.

Ideas are limited by language. Without symbols (or pictures) thought is impossible. In a given language the possible ideas are limited by the restrictions applied to the symbols. No language can exist without such restrictions. These restrictions limit the possible combinations and further limit the probable combinations. Because of combinatorial restrictions any symbol in an expression in a specific language applies restrictions to adjacent symbols and, in a degree inversely proportional to their proximity, to all other symbols in the expression. Thus the number of ideas that can be expressed in a given language is finite. Theoretically, complete unpredictability (i.e. a pure random source) implies maximum information in an entirely unintelligible message – (a confusing notion if there ever was one!) These are typical problems in existing Information Theory.

Contemporary computing machines are clumsy, inefficient, extremely primitive devices. The current emphasis on their importance in certain aspects of international relations, and to a lesser degree the advent of automation in private industry, will result in a continued acceleration of the program to improve them.

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